# Answer to Giuseppe Longo

## Synthetic Philosophy of Mathematics and Natural Sciences

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To begin with, I must thank Longo for his unusual generosity. It is extremely rare for someone to write such a lengthy commentary, while correctly highlighting a monograph's central ideas and, on top of that, to place them within a myriad of alternative mathematical considerations. And if that wasn't enough, Longo's subtle, critical and analytic scrutiny of what concerns my text *Synthetic Philosophy of Contemporary Mathematics* extends to the scope of a synthetic vision that encompasses the mathematics of biology. In many senses, Longo's remarks deserve to be understood then really as an appendix to my book which *opens* it, densely and broadly, to both the natural and mathematical worlds.

From Longo's many ideas, I would like to concentrate here on five broad themes that the wise Italian (yes: wise men are still among us!) emphasizes throughout his commentary: (1) The fundamental role played by the notions of *perspectivity*, freedom and purity in contemporary mathematics; (2) the *pendular equilibrium* required between processes of synthesis and analysis, between genericity and specificity, for the progress of knowledge; (3) the importance of the notion of action, both technically and epistemologically, for a thorough understanding of our world; (4) an emphasis on human *multidimensionality*, friction and contingency within the mathematical domain; (5) the importance of *gesture* and of 'metaphorization' for mathematical thinking. In what follows, I shall tackle these topics in order, and proceed in proximity with some of the various (quasi musical) variations in Longo's text.

1. Longo recognizes the central role that Grothendieck's plays in my monograph. Times have changed, and what Hilbert once was to Lautman, Grothendieck must be to us. Sixty years after his first great works, it is time that the community of philosophers of mathematics awakens. Longo highlights three great tendencies that Grothendieck's work has consolidated into pillars of contemporary mathematical thought. First of all, we have the construction of a relative mathematics, where a sophisticated *network of projections* allows us to couple, on the one side, mathematical structures and, on the other, explain the *back-and-forth* between these structures and the world. In Longo's words, "we only see perspectives, albeit coherent and profound ones; points of view on fragments of the world, we organize and make accessible small corners of it". Thus, the geometric network of perspectives according to Grothendieck is bound to a fascinating reincarnation of the idea of *freedom*, of *purity*; as Longo indicates: "Grothendieck proposed notions and structures of an intrinsic mathematical 'purity' free from any contingency requiring proof of invariance". By moving away from a framework of specific relations, and delving into the general, so-called *free objects* may in effect be projected in the whole categorical context that envelops them. For Longo, the iterated process of amplification, generalization, transformation and projection "is legitimate because, in this theoretical back-and-forth, our friction and action upon the world are real: the world resists, it says 'no', and channels our epistemic praxis, which is of an eminently *organizational* character, and it is always *active*". In this way projectivity, freedom, and purity become the ideal conditions for the emergence of mathematical activity, something that we could

directly relate to the mathematician's very *creativity*. In fact, abstraction, for Grothendieck, far from constituting a gratuitous act of ascent, is the very ground of invention. Proud to have provided over a thousand new definitions in mathematics, Grothendieck understands the realm of elevation, of projectivity, of purity, as the natural setting to become *exactly free*, distanced from the constraints of circumstance. Far from being a mere gratuitous artifice, abstraction thus becomes the *natural* environment so as to proceed without shackles. The rising tide that dissolves the nutshell turns out to be much more natural than the overwhelming and gimmicky hammer which shatters it into pieces. Grothendieck knows and declares himself heir to Galois: the V function that incorporates *all* root differences, and through which each one of them is represented (allowing the introduction of the Galois group transformations — in the original manuscript, the creative highpoint of the young genius) is an example of projectivity, freedom, and purity that Grothendieck has extrapolated to his fabulous techniques in schemas, topoi, and motifs.

2. The main objective of my book *Synthetic Philosophy of Contemporary Mathematics* consists in trying to open the dialog between mathematical philosophy and alternative perspectives, which are *non-dominant* in the field. I orient myself towards non-standard themes and processes, described Longo's terms as: "The realm of a plurality of Categories", "the constitution of concepts and structures", "a geometrical inspiration" which "makes us appreciate the structural sense of mathematical construction", a fight against a "ruinous disintegration of sense" proper of analytical perspectives, a study of "the difficult notion of border", an appraisal of "organizational gestures of correlated mathematical universes, correlated by a web of transformations". The established, constant task is that of seriously reading *Alice Through the Looking-*

*Glass* and of studying thus the *obverse* of concepts. The *opening* towards contemporary mathematics, synthesis, category theory, sheaf intuitionist logic, negativity (non-commutativity), deformation, Gromov's cloud — forms that are counterpoised to those of classical and modern mathematics: analysis, set-theory, classical logic, positivity, heirarchization, Hilbert's tree — encourages the exploration of underlooked *counterparts* in the history of philosophy and mathematics. The *initial task* consists then in *configuring the pendulum* of analysis and synthesis so that a thorough dialectic can be established between opposites. In what concerns my monograph, as Longo correctly notes, this is done by underlining diverse back-and-forths that make up a *general Galois theory* with multiple levels of linkage: languages and geometries, proof and structural synthesis analysis, symmetries and ruptures, specificities and genericities, 'smoothnesses' and frictions, universalization and contingency, cosmos and humanity. Now, once the initial task of *situating* alternative perspectives is overcome, the truly important task in the future will be that of creating entire new branches of thought bound to the *mediation* of analysis and synthesis. With Roberto Perry, we call horosis (horotic transformation) this mediation (from *horos*, border), and our great project for the next five years (2015-2020) consists in providing *a systematic organization of the mathematical, philosophical and artistic constitution of horosis in the 19th and 20th Centuries*. Just to give one example of the enormous richness at stake, if the Greek tension between the One and the Many becomes incarnated in Cantor's definition of the set as that which is *analytically one and many at the same time* (*ramified tree*: level n+1 = One = set; level n = Many = elements) and also in category theory through the motto (Yoneda's lemma) that objects are in reality their representable functors (synthetically one and many at the same time: an object is identified with the crown or aura of its morphisms), a similar situation should obtain in turn for the horotic understanding of the One and the Many. It is worth mentioning that the *axioms* of set-theory (particularly the axioms of separation, pairs, union, and powerset) are immediate consequences of the analytic definition of the set, just as the *axioms* of category theory are the immediate consequences of the synthetic view of representable functors. Once we obtain a *well defined metaphor* of the objects pertaining to horosis (neighborhood, borders, etc.), the natural axioms for a general border theory (much more fundamental than Thom's cobordism) should emerge, as the new Century will quite probably demand. Just to give one additional variation on this subject — as suggested by Longo when referring to Gödel's incompleteness theorem, orienting us towards searching "*only* by looking closely to its proof" — it is surprising how the first paragraphs of Gödel's Doctoral Thesis (1929-1930), in parallel to his completeness theory (for first-order classical logic), already *explicitly* indicate the possibility of his incompleteness theorem (for arithmetic) for *natural intuitionist* reasons. There have been many betrayals of Gödel throughout the twentieth Century (perhaps only Georg Kreisel has understood him in depth) and the *disappearance* (imputable to Hahn) of the initial paragraphs of the Thesis in the later article (1930) is one of the many moments in which a *fully triadic* Gödel (*at the same time* intuitionist, logicist and formalist) has been conveniently reduced and simplified by questionable 'philosophers' of mathematics. Gödel's extraordinary *phantasmagoria* (magnificently studied by Pierre Cassou-Noguès) lives in fact within an *essential border* between saturation (completion) and compression (recursion, ground of incompleteness), classicism and intuitionism, linguistic minimization ($V=L$, $c \aleph_1$) and harmonic maximization ($Compl(V)$, $c = \aleph_2$). In fact, all of Gödel's work can be understood as a f*ascinating incarnation of the horos*. Gödel's pendulum, oscillating between satisfiability and refutability, numeralization and representation, ennumerability and transfinitude, classical and intuitionist translations, analytic irradiation and phenomenological irradiation, and between the living and the dead, shows us the complexity of a character and of an oeuvre that can only be understood in depth by *shattering* our habitual categories and prejudices. In this sense, perhaps only novel ways of doing philosophy will be able to understand Gödel's approach in the twenty-first century (remember his famous motto: "Philosophy today is, at best, at the point where Babylonian mathematics was").

3. An outstanding point in Longo's commentary concerns his subtle consideration of the place that *action* holds, both for processes of knowledge in general, and for mathematics in particular. For Longo, "science is not a testimony of, but an *action upon* the world, aimed at organizing it and giving meaning to it". In fact, action is indissolubly bound to the *projective exercise of reason* and to the *pendular sway of knowledge*. Action places us in warp-zones of relations with respect to 'something' (Peircean secondness) and constricts us to finding mediations that refine these warp-zones (Peircean thirdness). Longo highlights "the constitution of invariants that lies at the heart of the construction of (physic-mathematical) knowledge, in continuity with *action* upon the world, yet not in the world itself". Action imposes a *distance* — a constitutive element of intelligence, according to Aby Warburg: [t]he conscious creation of a distance between oneself and the exterior world may be considered as the foundational act of human civilization" (this is the famous first line from the *Introduction* to the *Mnemosyne Atlas* [1924-1929]) — from which originates, in Longo's words, "the mediation or interface between mathematics and the world". We find ourselves then in a fully relational domain, relative, projective, free, which of course takes us back to Grothendieck, but that is clearly expressed in

many past thinkers, from Novalis to Cassirer, through Peirce, Valery, Warburg, Florenski, Benjamin, and so many other modern masters, attentive as they were to exploring the relational networks of understanding and of sensibility. The analytical inquisitions, a thousand times futile, concerning the ontological and epistemological status of numbers and sets have little to do with the activity of mathematics on the world. Much more real and coherent are the investigations of the last decades (Petitot, Berthoz, Citti and Sarti, Longo, etc.) that highlight a *protogeometry* prior to number and to language in what concerns the essential acts of our understanding, something that for Longo is summarized in "a plurality of praxis from which to distill an invariant in memory and then produce (in language) number, in order to stabilize a concept resulting from a practical invariance with a long evolutionary history". The *primordial protogeometry* of human imagination is very well expressed in some of the greatest works of 20th Century literary fiction, such Robert Musil's *Man Without Qualities* (1921-1942), or Hermann Broch's *The Sleepwalkers* (1930-1932). Broch, who studied with Gödel at the University of Vienna, at the end of the 1920s, explains how "the internal relations of mathematics are projected into a logical sphere and then can be mirrored in turn as reflexive projections, projections of projections; it is also possible to imagine an infinite multiplicity of 'relations of relations' between the mathematical and logical spheres" (Manuscript, "On Syntactical and Cognitive Units", not dated). The active, projective, and relational character of mathematics (and of its logical sub-fragments) *forces a continuous knowledge, plastic, approximate*, and is at the same time linked to the fact that no real measure in the world is punctual, governed by classical or entirely determined laws — as Longo indicates, "*measure* necessarily is, because of physical principles, an *interval*". The natural logical of continuity, of plasticity, of intervals must be in truth topological.

The logic of Peirce's existential graphs, intuitionist logic, the logic of complex variables, categorical logic, and the logic of sheaves — topics extensively examined in my *Synthetic Philosophy of Contemporary Mathematics* — become thus new (topo)logical pillars for the understanding of a multivalent, multidimensional, complex reality. Parallel to this, Longo highlights other trends in the theory of computation, in the compression of neuronal organizations, and in interfaces with biology, all of which would require important advances towards a sort of *logical smoothening* of the kind that analytic philosophy utterly ignores.

4. One of the most beautiful emphases in Longo's remarks is to be found in what we may call his *ode to humanity*. The Italian wise man unleashes his poetic drive in various moments, as his true *love* for human intelligence surges forth. Longo teaches us that mathematics is especially beautiful owing precisely to the profound philosophical obstacles it must overcome: its friction with the world, its contingency, its historical evolution, its swaying creativity. Far from indulging in those artificial dissections and false dialogs common among the sect of so-called 'analytic' 'philosophers' of mathematics, Longo incarnates in the human — in an almost Nietzschean ode to the most human — the extraordinary force of mathematical imagination. We face the exact opposite situation to that found in the *spectacular countersense* offered in the *Oxford Handbook of Philosophy of Mathematics and Logic* (2005), where *not a single mention of real mathematics appears*, and where man, history, the world, creativity, beauty and metaphysics disappear under the arid scalpels of those surgeons who have sought to eliminate, in their disparate linguistic investigations, the soul and heart of deep mathematical thought (Galois, Riemann, Poincaré, Grothendieck, Lawvere, Connes, Gromov, etc). For Longo, "[a] first way of being in the world and of constructing

the intelligibility of the world with other disciplines, indeed is to appreciate its 'dimensionality', in the entire semantic richness of the world". A *true comprehension* of the semantic multidimensionality of the world and, in particular, of the spatial and structural multidimensionality of mathematics, is without doubt one of the greatest tasks facing the twenty-first century. In order to attune itself to its outside, philosophy must escape from Babylonian times (Gödel); it must become open to the myriad of fabulous mathematical techniques invented in the twentieth Century, *liberate* its conceptual and imaginative flight, and project its inventive arsenal towards the thousand forms in which contemporary culture shapes and submerges us. In times of unparalleled scientific invention, but also of fascinating artistic explorations — science and art being both amputated under analytic perspectives — we *demand* new odes to humanity, as that delivered to us by Longo. According to Longo, the process of invention "assumes an historicity that serves to highlight the sense and the relationship of mathematics vis-à-vis the real: mathematics (where it does work) and has meaning because it is constituted through a human — all too human — praxis". And he reiterates: "I would go as far as to say that mathematics helps us to construct objectivity precisely *because* it is contingent, the result of the 'history' of a real friction with the world". The extraordinary accuracy of Longo's argument incites us to marvel at what has been called, in manifold ways, the *miracle* of mathematics: its *horotic status* between the necessary and contingent, the ideal and real, the theoretical and the applicable. The force behind Socratic *surprise*, and the true love for mathematical philosophy, lies precisely at the e*dge of this abyss*. The best definition of mathematics is possibly found in variations around this Gödelian *border*, so characteristic, between the universal and the particular, between what Longo calls "the *genericity* of objects and the specificity of their trajectories". Far from the allegedly deductive character that defines mathematics (Russell's banal definition), it is rather the *forms of deduction* it deploys (the *interior* of proofs, according to Longo) and its forms of retroduction (Peircean *abduction*) that are fundamental. In truth, one of the most provocative contemporary investigations on mathematical foundations, Voevodsky's homotopy type theory (HoTT, 2006-present), seriously considers the idea of observing the interior of proof procedures, and proposes to take as primitive the trajectories of objects, *before* the objects themselves. We have here in truth another sophisticated expression of a *subjacent protogeometry,* one which could transform, within the next decade, our understanding of the mathematical world.

5. The creative impulse of metaphors in mathematics ought never to be underestimated. Galois' *theory of ambiguity*, where the indiscernibility (obstruction) of the roots from the base field becomes conceptually *inverted*, and opens the way to its transformations (transit) codified in the *Galois group*; Riemann's negative harmony, where the multivalence (obstruction) in the logarithmic function of a complex variable becomes in turn inverted and opens the way to its extension (transit) over the relevant Riemann surface; Poincaré's structural networks, where Poincaré's sphere serves as a counterexample (obstruction) to the attempt of characterizing homologically the sphere $S^3 \subseteq R^4$ and gives way to the possibility (transit) of characterizing it homotopically (Poincaré's conjecture), these are all examples of great 'metaphorizations' that combine *general imaginative flight* and precise, particular techniques. Bound to the power of metaphor lies the primordial dominion of the mathematical *gesture*, oftentimes underlined by Longo, in reference to "that masterpiece that is Châtelet's L*es Enjeux du Mobile*", an appraisal with regards to which we are in full agreement. We must note here the excep-

tional enterprise advanced in this moment by Guerino Mazzola who, after his monumental *The Topos of Music* (2002), has opened a gigantic, alternative path in his *La verite du beau dans la musique* (2007). We are dealing with nothing less than the attempt to establish a thorough pendular sway — a Galois adjunction — between score and interpretation, where the study of interpretative gestures is realized through a sophisticated homotopy theory within very general categorical frames. With bravery, by means of the anti-postmodern motto "the truth of the beautiful", Mazzola expresses his vocation to approach the archetypes of musical invention.

6. It is about time for the philosophy of mathematics to begin to redraw the map of the great contributions in the discipline bestowed to us in the 20th Century where, it must be said, the place of France acquires an ever-growing projection. Lautman, Châtelet, Petitot, or Badiou, just to mention a few particularly original views, go incomparably farther than their English-speaking counterparts, although the latter go by the more "popular" names of Quine, Putnam, Field or Maddy. The reason is simple: the former observe mathematics in action (Hilbert, Riemann, Grassmann, Hamilton, Thom, Ehresmann, Cohen, Grothendieck, as outstanding figures), while the latter only observe fragments from logic and crosslink references, associated to secondary literature. The example of Giuseppe Longo, so attentive to the Italian and French traditions in the philosophy of science, as well as the technical advances achieved in recursion theory and in the biological work of his English-speaking colleagues, must serve as a guide to shatter comfortable frameworks. Following Benjamin's typography in *Passages de Paris*, the philosophy of mathematics must "Awaken".