Mathematics Lessons Using Learner-Generated Examples

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Final Project Report

March 20, 2010

Acknowledgements

I would like to thank Dr. Janet McVittie, Graduate Coordinator of the Department of Curriculum Studies, for her support and encouragement over the past three years. Her advice, patience, and encouragement have been an invaluable part of this endeavor.

Egan Chernoff, who has provided me with many ideas, support, insights, and encouragement to bring this project to fruition. His knowledge and enthusiasm in the teaching of mathematics is inspirational. Thank you Egan.

The Faculty members of the Department of Curriculum Studies have provided me with many learning opportunities and their professionalism and teaching is commendable.

My family, Karen, Alex, and Kian, have been a huge support in nudging me along to complete my studies. Their patience and moral support has been greatly appreciated.

The Living Sky School Division, for being very generous in granting monies to reimburse the tuition fees at the U of S. Thank you.

Thank you to the students of my school for their cooperation and interest in learning mathematics.

Finally, thanks to the rest of my family, neighbours, and colleagues for their loyalty and friendship.

Abstract

It is widely accepted that mathematics is learned by engaging in activities that invite learners to be active participants of the knowledge. Teachers strive to provide best teaching practices but sometimes the learner does not understand what the teacher intends to teach. Learner–generated examples (LGEs) is a learning/teaching strategy in which learners are invited to construct their own examples of objects such as numbers, diagrams, equations, or other mathematical entities as a means to develop a more enriched understanding of mathematical concepts. This paper describes the characteristics of LGEs and as a final product, it provides secondary teachers of mathematics with a compilation of lesson plans suitable for their daily practice.

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Preface

Although reform in mathematics has been advocated by the NCTM (1989) for over two decades, many teachers have been slow to change their approaches. In a content-driven curriculum, it is easy to get stuck on the educational treadmill of *covering the course.* A common topic of conversation among math teachers is the unit they are on and when they expect to finish the course. It is as if they are in a race and the first one to complete all the units is the winner. Perhaps mathematics teachers are reluctant to change, because as students, their learning styles allowed them to be successful in a traditional system that emphasized a direct teaching approach. In essence, some teachers do not see a problem with the status quo of current pedagogical practices. Yet in conversation with these teachers, many are not happy with their work or the lack of success of their students.

With the implementation of the new Western and Northern Canadian Protocol (WNCP), there is a push to adopt the student–centered philosophy recommended in the new curriculum guides. As well, there are fewer outcomes and teachers will have more time to provide exploratory activities that engage and encourage critical thinking. Now is an opportune time to convince teachers to use instructional strategies that encourage students to reflect and make connections with concepts so that the mathematics they are supposed to be learning starts to make sense. Hopefully in Saskatchewan we have reached a tipping point in mathematics education, and teachers are willing to experiment and play around with unfamiliar approaches as they attempt to implement the new curricula.

About a year ago I began reading about one of these unfamiliar approaches known as learner-generated examples, or LGEs. This classroom practice involves teachers inviting learners to construct their own examples of mathematical topics and by doing so they gain a better understanding of what is being taught. Teachers often provide examples to learners as a means to communicate and engage in mathematical discourse, however, in using learner-generated examples, students attempt to construct their own examples of mathematical objects under given conditions. This is a fundamental shift for experienced teachers like myself in which following specific examples is the norm. Changing teaching practices is not an easy task and using LGEs is no exception. As I began tinkering with this strategy in my classroom practice I began to start to appreciate the impact it can have on learning. It is uncomfortable using new teaching strategies, especially when it appears one is giving up some control of the classroom, but in my experiences with LGEs, there are many benefits and insights to be gained by the teacher and student.

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Introduction

The new age of Math Reform is dawning. Although reform in mathematics tends to move slowly, there is a shift towards more classroom practices in which teachers are providing students with many opportunities to solve complex and interesting problems, to read, write, and discuss mathematics, and to test their personally constructed mathematical ideas so that they can draw their own conclusions. This is in contrast to the “Old School” or traditional methods that consist of mimicking procedures and using simple algorithms. Old School methods result in merely a vague sense of comprehension. As Michael Battista (1999) puts it, "for most students, school mathematics is an endless sequence of memorizing and forgetting facts and procedures that make little sense to them" (p. 423).

As an experienced teacher in a small rural high school, change in mathematics pedagogy can be a slow process. Generally, teachers tend to teach as they have been taught. With the current implementation of the new WNCP Math curricula, a constructivist teaching approach is recommended (NCTM). Constructivism is a learning theory which argues that learners are active in constructing their understanding. They do not passively receive or absorb information (Simon and Schifter, 1993). Armstrong (2003) described this learning theory in which "each student builds a personal set of knowledge which evolves as he or she relates to new information and puts it together with previously held understandings" (p. 111). One of the problems teachers have with a constructivist approach is that they are simply not sure of what they should be doing.

In an attempt to provide constructive activities that engage and challenge the learner, this project will delve into a teaching strategy known as learner-generated examples, hereafter, to be referred to as LGEs. Two of the key researchers of this strategy are Anne Watson and John Mason. In their book, Watson and Mason (2005) write extensively on the benefits and potential of having learners create their own examples. Other educators (e.g. Hazzan, Zazkis) have also found that examples generated by learners serve as a powerful pedagogical tool for enhancing the learning of mathematics.

This project promotes the use of LGEs as a mathematics learning/teaching strategy within the community of mathematics teachers. This report and the lessons included will provide math teachers an opportunity to learn about LGEs so that they have one more teaching tool to engage their students. The project involves the development of an instructional booklet that contains lessons aligned with the new WNCP Mathematics curriculum for grades 9-12. As teachers begin the challenge of implementing the new curricula with its emphasis on student-centered activities, these lessons can be a useful resource. Teachers who want to experiment with the lessons can use the teacher notes as a guide, and if they wish to learn more in detail about LGEs they can access this report for a deeper analysis. The lessons will be available on a wiki website or teachers can contact the author if they are interested in a printed copy.

Literature Review

Early research on examples

George Polya (1981) used long sequences of exercises to build generalizations from a simple idea. He instructed students to "devise some problems similar to, but different from, the problems proposed in this chapter – especially such problems as you can solve" (p. 98). Polya also delved into the use of examples to illustrate how extreme ones can create unusual results. Michener (1978) used reference examples to test out conjectures and to illustrate how proofs of theorems work. She advocated for an active learner who gains understanding by examining and fiddling around with examples rather than the passive learner who does little and waits for understanding to happen.

Charles (1980) conducted a study among teachers to determine the effects of *exemplification moves* in facilitating the acquisition of mathematical concepts. He concluded that the use of examples and non-examples did impact learning. Mason and Pimm (1984) explored the notion of a generic example and proposed that there are "inherent difficulties in mathematical expressions of generality and their relation to the particular" (p. 287). Sowder (1980) stated that passive acceptance of given examples does not result in deep understanding of a concept. He recommended that learners need to construct their own examples prior to stating formal definitions.

Exploring Example Spaces

The positive effects of having students create their own questions can be motivational and beneficial to better understanding concepts (Bell and Swan, 1995). Watson and Mason

(2005) described exemplification as "any situation in which something specific is offered to represent a general class with which the learner is to become familiar – a particular case of a generality" (p. 4). An example of exemplification might be a drawing of a particular rectangle to accompany a definition of rectangles given in a textbook. The learner is able to have an understanding of rectangles, the general, by seeing a particular rectangle. A typical LGE might be constructed using this sample activity: *Construct a parallelogram with at least 2 right angles*.

This student-centered approach forces the learner to explore concepts and make connections with knowledge that may have seemed unrelated. The remarkable feature of this task is the role reversal of teacher and learner. Traditionally, the teacher provided these data and the learner computed the mode, median, and the mean. A key feature of this approach is that the examples come from a range of possibilities and it is important that teachers and learners try to become aware of the potential and limitations of their experiences with them. Watson and Mason (2005) formulated the notion of a *personal example space* to describe the uniqueness of using LGEs. LGEs give learners a chance to explore, arrange, and gain fluency with these spaces which can be described as individual and situational:

> dependent on the knowledge, multiplicity of experience, and predisposition of the learner but it is also framed by the wording of the prompt, who is making it, under what circumstances; different learners may respond with different examples in the same learning environment, and the same learner may respond differently in different situations. (p. 56)

According to Watson and Mason (2005), it appears that all learners are capable of generating examples and as they do so their confidence grows rapidly. Being aware of the differences in personal example spaces can guide or influence the kinds of examples employed in the classroom. Conventional example spaces found in textbooks may limit the development of learners' personal example spaces. Zazkis and Chernoff (2008) introduced the notions of pivotal example and bridging example to demonstrate that the convincing power of counterexamples may depend on the extent to which they are in accord with an individual's personal example space. They concluded that "a strategic example that is pivotal for one learner in a given situation may not be helpful to another learner or in another situation" (p. 207).

Extending Example Spaces

As presented, prompting learners to construct their own examples is an effective strategy for shifting from a teacher-centered to a learner-centered pedagogical environment. An important pedagogical aim of LGEs is to promote awareness of the range of possibilities and to engage the learner in the deeper understanding of the concepts. Creation of an example is a complex task compared to simply executing a standard task. Learners who generate their own examples as a strategy of learning are more likely to understand new concepts (Hazzan & Zazkis, 1999). As learners create their own examples they experience the assembly of a space of objects and they can "recognize new relationships, enjoy new meanings and personal understandings" (Bills, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006).

Zazkis and Leikin (2008) asked undergraduates with strong math backgrounds to provide many examples of definitions of a square. Some participants had trouble providing appropriate definitions hence revealing their inability to distinguish between necessary and sufficient conditions. As well, some of the standard definitions that are used in early grades may create misconceptions in later years. This study demonstrated that LGEs can serve as a lens through which teachers can view comprehension difficulties of mathematical concepts. In this way, teachers are able to monitor and assess student learning so as to be able to modify their teaching.

Through construction of examples, learners become aware of dimensions of possible variation. Surprisingly, even experts in mathematics may only see a fragment of a large example space. Spaces are often dominated by strong images, some of which may be universal. The experience of constructing examples for oneself may contribute to the ability to trigger richer example spaces in future situations (Bills et al., 2006). In activities in which students are given choices, there is a tendency to avoid numbers that may not work out. LGEs allow individuals to take risks and explore so that the learner gains the confidence to tinker and search for the general class of examples involved (Hazzan & Zazkis, 1997).

Learning New Concepts with LGEs

Could students learn a completely new concept using learning generated examples? In a research study by Dahlberg and Housman (1997), undergraduates were given a definition of an unfamiliar concept and asked to generate examples using the definition. Students given only the correct definition learned a significant amount and actually showed more comprehension, compared to students using other strategies such as memorization or decomposition. The researchers suggested that "it may be beneficial to introduce students to new concepts by requiring them to generate their own examples or have them verify and work with instances of a concept before providing them with examples and commentary" (p. 297).

In a recent study at an English comprehensive school, Watson and Shipman (2008) did a two- part study involving high and low achieving students. In the first study, they introduced the high achieving Year 9 class to the idea of irrational numbers by asking them to generate examples involving roots and the way that they behave. For comparison purposes, a second group of high achieving students also learned about the same concepts of irrational numbers using a rule–based approach focusing on definition, technique, memorization, and application in given questions. Not only did students using LGEs observe numerical patterns generated in sequences using different examples, but they were able to make conjectures regarding the relationships connecting the variables within examples. The researchers concluded that "their experiences can be organized in such a way that shifts of understanding take place as a result of learners' own actions, including mental acts of organizational reflection on self-generated examples and example spaces" (p. 108). According to Shipman, the parallel group could cope with familiar questions but struggled with more complicated problems.

In the second study of 16 year olds, Shipman prompted students to generate examples of right triangles and find the relationships of primary trigonometric ratios. As a consequence of the class activity and discussions among themselves, the class managed to successfully solve 30 typical examination questions designed for higher-achieving learners. Shipman compared these results to another group of 16 year olds who had been previously taught these trigonometric ideas 2 years earlier. These students were revising the exact same concepts in preparation for the upcoming high-stakes examination. This group was unable, in a group discussion, to achieve a comparable level of competencies with unfamiliar questions. According to Shipman, this group lacked the underlying knowledge and intuitions required to do well on the questions. Of particular note was that such shifts were visible in a previously low achieving group of students which counters criticism of LGEs as a strategy exclusive to advanced students.

A Summary/Framework

Despite the research suggesting LGEs is a useful learning/teaching strategy, this is not a typical classroom practice in Saskatchewan schools. Over the past several months the routine of introducing LGEs in my classes has given me the confidence to implement them whenever the opportunity arises. The lessons that have been produced are based on themes put forth by Watson and Mason (2005) in which they summarize LGEs into five major categories: (1) the role of LGEs to develop example spaces (what kinds of numbers or objects are there ?); (2) extending example spaces and becoming more aware of constraints; (3) understanding the structure of mathematical concepts; (4) exploring example spaces and experiencing patterns or generality; and (5) learning a new concept using LGEs as an investigative strategy. As with any new teaching tool, becoming confident at using LGEs in the classroom takes time, perseverance, and a bit of risk-taking.

Methodology

Early stages

Lesson plans using LGEs for the instructional manual were created for grades 9-12 between the months of December 2009 and February 2010. The lesson plans for grades 10-12 were designed to meet the goals of the current provincial curriculum, however the intention was to also choose topics to fit the new WNCP curriculum which begins its implementation in 2010-2011. The math lessons for Grade 9 did align with the new WNCP since this course is already in its first year of implementation. There was a slight tendency to focus on the Grade 9 course because the new curriculum guides often suggested investigative activities that dovetailed with LGEs.

Resources

Several textbooks written by Burt Thiessen provided some of the mathematical content of the lessons. The teaching resources of *The Math Makes Sense* (Pearson Education, 2009) program also provided ideas that fitted well with LGEs. *Mathematics as a Constructive Activity: Learners Generating Examples*, (Watson and Mason, 2005) was an excellent resource and several studies described in the literature review provided the spark to construct the lessons. Some of the lessons originated from learning/teaching experiences in the classroom.

Ethics

Ethics approval was applied for and granted by the University of Saskatchewan's Behavioural Research Board (Beh-ReB). These documents are found in Appendix A . Consent forms for the students and parents are found in Appendices B and C. Sixteen Grade 12 students responded to question prompts written at the end of the lessons and these prompts are found in Appendix D. At the end of the semester when marks had been submitted, the grade 12 students were invited to share these comments with the author. Parental and student consent forms were collected from those students who wished to be a part of the project. In total, five students returned their consent forms and comments. These comments are included in the next section of this report under Discussion.

Obstacles and design dilemmas

The creation of the lessons resembled a slow evolving process. Typical of any new strategy, there was a learning curve and lessons that seemed fairly adequate were later considered to be not worthwhile or not quite fitting an LGE approach. In fact, this realization occurred after a discussion with my supervisor who suggested that the lessons be analyzed carefully for LGE content and overall teacher usability. The analysis indicated that the lessons had a constructivist orientation but not much of a real LGE flavor. There were parts of the lessons that involved LGEs but nothing substantial. Finally, the idea of a framework emerged which involved classifying the lessons according to five major uses or categories.

In the development of the lessons, a dilemma arose as to how much detail needed to be included in the Teacher Notes of the lesson plans. Rather than omit some of the key theoretical underpinnings of LGEs and the rationale for using them, the design of having a chapter that dealt with an in-depth analysis of the five main categories and their corresponding lessons seemed a logical way to proceed. This chapter analysis provides an in-depth look at the characteristics of LGEs for interested educators who wish to probe more deeply into the pedagogy of learner-generated examples. It represents an extension of the product of this project.

Final product

Hence, this project has been designed to consist of two major sections. One section consists of a detailed analysis of five particular lessons using learning generated examples (LGEs). These lessons have been specifically selected because they are representative of the five major categories of LGEs. In keeping with exemplification, we are using five examples to exemplify the concept of LGEs. The second section consists of the end product of the project which has a set of 14 mathematics lessons using LGEs and are found in Appendix E. The lessons, designed for students in grades 9-12, contain a Teacher Notes section, and provide background knowledge or information that may be helpful in delivering the lesson. The Student Notes can be found in Appendix F and they provide the student with a written account of the learning material for the lesson.

A Comprehensive Analysis of Learner- Generated Examples

Introduction

This project strives to describe and demonstrate a teaching strategy in which students generate examples to enhance their understanding of mathematical concepts. Although production of examples has almost always been the responsibilities of teachers and textbooks, there is a trend to invite learners to create the examples for the lesson. The following analyses describe some of the main characteristics and uses of learner-generated examples (LGEs). Each topic provides an in-depth look at particular aspects of LGEs within the context of a mathematics lesson. An abbreviated version of these lessons without the detailed analysis can be found in the lesson booklet in Appendices E and F.

Each lesson consists of an introduction of the lesson and some of the key features of the LGEs. The first chapter contains a general overview of learner-generated examples while the others tend to focus on one or two particular areas. These areas or general uses can be divided into five main categories but sometimes there is not a definite demarcation. These uses include: (1) the role of LGEs to develop example spaces (what kinds of numbers or objects are there?); (2) extending example spaces and becoming more aware of constraints; (3) understanding the structure of mathematical concepts; (4) exploring example spaces and experiencing patterns or generality; and (5) learning a new concept using LGEs as an investigative strategy. In keeping with the philosophy of the new curriculum, these lessons invite students to be active learners and to provide the raw material or data needed to understand the intended outcomes.

Each lesson plan includes the course, the strand, and the outcome. The Teacher Notes section provides the theory and rationale for using the LGEs to inform and guide the teacher. The Student Notes represent a structured plan with the tasks recommended. They are a means for the student to follow the sequence of tasks and provide a written summary of the student's thinking and work. The plan is only a suggestion to be used at the discretion of the teacher. As with most lesson plans, modifications and adjustments may be beneficial.

Category 1: Characteristics of LGEs and Developing Example Spaces

One of the main goals of this unit on rational numbers is for learners to have a better understanding of where numbers belong on a number line and to be more aware of the relationships between integers, fractions, decimals and mixed numbers. An easy and effective way to introduce LGEs is to simply ask students to give an example and to allow the question to be open-ended. Students may initially be uncomfortable with this kind of questioning because they may suspect the teacher has a specific concept in mind. Students may feel inhibited because perhaps their number is not good enough. One of the main characteristic of LGEs is that exemplification or the creating of examples is individual and situational. It is important that teachers recognize the personal nature of sharing examples and that those examples might depend on students' expectations of the task, as well as their knowledge, experiences and their environment.

Teacher Notes - Math 9 - Number Strand
Outcome 9.2: Demonstrate an understanding of rational numbers including comparing and ordering.

Task #1: *Think of a number* ______ *Think of another* ______ *And another* ______ *Share your thoughts with a classmate.*

Students will often think of the first number that comes to mind but asking for more examples tends to motivate students to think of more interesting examples and they start to explore other possible numbers. After several months of implementing LGEs, one of the benefits of exemplification is that it often exposes a range of possibilities or a class of examples that may be useful to other learners. It is important that students have the chance to share their examples in some kind of public manner such as a classroom discussion or on a blackboard. In this case, students could share with a classmate and then put their numbers on the blackboard. The numbers will then create a discussion among the students and since the topic is rational numbers the teacher can initiate a discussion to help students decide which ones fit the definition. In my experience with this task, students often create whole numbers and will not think of negative numbers, fractions, or decimals until they see one show up on the board. The next example demonstrates how to take students to this larger set of numbers if necessary.

Task #2: *Think of a rational number that lies between 1 and 2 on the number line.*

By placing constraints on the examples, students are forced to leave their example space of whole numbers and start looking for fractions or decimals. As students are trying to think of examples you may want to think of the kitchen cupboard suggested by Watson and Mason (2005) as a spatial metaphor to represent example space. In it one finds familiar examples clustered at the front but further back are those items less utilized. It is convenient to find conventional examples that dominate textbooks or age-old classroom examples but if challenged to generate new or unusual examples the learner has to move "stuff" around to see what is available.

With the implementation of LGEs in the classroom, the role of the teacher changes since it is the students providing the examples for the lesson. However, teachers need to be aware of the key outcomes intended in the lesson and if they want students to review or introduce certain concepts they need to be prepared to use the examples generated to initiate discussion and thinking among the students. In the above task, students may provide examples of decimals but may not make connections between fractions and decimals. Task #3 can help them investigate the relationships.

Task #3: *Think of a fraction and then change it to a decimal.* _________ *Think of another one and change it to a decimal.* _________ *And another one* _________

The teacher may need to remind the class how to change fractions to decimals by division but most likely some grade 9 students will remember this concept. However, we know that a good teacher must be careful not to make too many assumptions. The use of LGEs allows the exploration of many kinds of fractions and corresponding decimals. The examples can be shared with the class on the board by having the students create 2 lists of terminating and repeating decimals. Now that students have created numbers in the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$, the teacher can reinforce the term **rational number**. Note that although the teacher could investigate division by 0, this concept requires a fair amount of time to address properly.

Task #4: *Create a fraction.* ______ *Create a smaller one.* ______ *And a smaller one.* _______

Learner-generated tasks can lead students to start making generalizations. As students make the fraction smaller they may decide to increase the size of the denominator or decrease the size of the numerator. Of course, they may change the fraction completely. Nevertheless, they may see a pattern as the denominator increases.

Task #5: *Think of a proper fraction that has a numerator of 1. Now think of 4 more proper fractions that continue to become progressively smaller.______ ______ ______ _______ Find a proper fraction that is extremely small compared to the others.* ____ *What kind of pattern have you observed?* ____

Similar to Task #3, asking for more examples will often force students to explore and gain a better understanding of the mathematical structures involved. In this task you are restricting the example space and stimulating the learner to come up with the examples. This task could lead to a discussion of limits if a teacher wanted to enrich the conversation. How small can we make this fraction? What number are we starting to approach?

Task #6: *Draw a picture that illustrates this concept.*

When students are asked to make a picture or drawing the wide range of constructions is both interesting and can generate plenty of discussion. Students tend to enjoy creating images, and LGEs give students an opportunity to express their mathematical thoughts. Many students like pie diagrams or rectangular shapes, but they should be encouraged to use any pictorial form.

Task #7: ***(a)*** *Think of 2 fractions that lie between* $\frac{1}{5}$ *and* $\frac{4}{5}$. ______, ______ *(b) Think of a fraction that lies between* $\frac{1}{5}$ *and* $\frac{3}{5}$. ______*(c) Think of a fraction that lies between* $\frac{1}{5}$ *and* $\frac{2}{5}$. ______*(d) Think of 3 fractions that lie between* $\frac{1}{5}$ *and* $\frac{2}{5}$.______, ______, ______.

Initially, students' previous knowledge will allow them to do parts a and b; however they may experience problems on the subsequent tasks. Will students be able to complete these tasks? Giving challenges to students is one of the benefits of LGEs. It allows them to become more aware of mathematical structures and connections that need to be made. If they decide to create equivalent fractions and use 10 parts, their efforts are in vain. However, by continuing in this process and creating larger parts, they will be rewarded. Students seem to enjoy this kind of exploration especially if teachers allow students to work together.

What happens if students cannot think of an example? In most cases, students will create an appropriate example but in the case this does not occur, teacher intervention may be needed to give students some insights into what may be required to generate the example. Sometimes this may be in the form of a prompt or perhaps by putting on a constraint. In the above case, it may involve the teacher drawing a number line, placing the fractions, and then asking the class for suggestions.

Task #8*: (a) Think of a number that is close to but less than 1.* ______ *(b) Now find a number that is even closer.* ______ *(c)Now find the largest number possible that is close to 1 using only the digits 2, 5, 7, and 8* ______

The use of LGEs reinforces the concept of place value as students start to fully appreciate what kind of number is required to meet the criteria. This kind of question is much better than the typical textbook question "is 0.9 less than 1?" In part(b), the student is challenged to search for another number and one of the claims of using LGEs is that students can gain new knowledge as they attempt to generate an example. As well, their understanding of the concept is more enriched. Asking students to create examples that lie just outside their realm of knowledge gives them a better understanding of place value and the nature of decimals. In part(c) there are many possible examples, but the student is seeking for the one example that is correct. Although constraints can reduce freedom of choice, they sometimes open up new example spaces and they can provide teachers with knowledge of students' misunderstandings.

Task #9: *Think of a number that is close to 3 without using the number 3 or the number 9.*

See the above paragraph for comments.

Category 2: Extending Example Spaces

Is it possible for teachers to develop or expand the example space of a student, or for that matter, the example space of the teacher? When students are asked for examples sometimes learners are only able to give a fragment of examples even though a large example space may exist. Such examples may be influenced by recent mathematical experiences, memory or even the image from a textbook. If a student is asked to give an example of a number they may only think of a small positive number. If it happens that a friend suggests a fraction suddenly a whole class of numbers opens up and comments can be heard such as “Oh, I didn’t think of that.” One of the roles of LGEs is to prompt or provoke students to engage in activities that makes them reorganize what they knew so that it fits the kind of example or object being sought. Bereiter and Scardamalia (1987) described this kind of constructive activity as knowledge-transformation rather than simply knowledge-telling.

There are certain examples that have been used in countless numbers of textbooks for generations such that they have become dominant figures to represent a certain concept. Fischbein (1993) coined the term figural concepts to represent these images that tend to take on unintended attributes. For example, most triangles are acute and have one edge parallel to the bottom of the page, rectangles are the most familiar quadrilaterals, and the most common irrational root is $\sqrt{2}$. The problem with these conventional examples is that learners can have misconceptions about the features of the figure. Some learners have named the first diagram a stick (Sfard, 2001) and excluded it as a triangle since it did not fit their image of what a typical triangle should look like. The second diagram is a square, but the third diagram becomes a diamond because the bottom side is not parallel to the edge of the page.

Figure 1

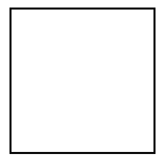


Figure 2

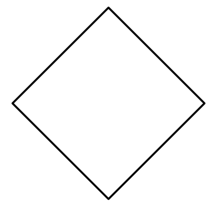


Figure 3

An important role of LGE tasks is to invite learners to construct examples that broaden their range of permissible change in the kinds of images they associate with these concepts. As students broaden their experiences with these concepts they will be less likely to be misled or have misconceptions of a topic. If a teacher asks the student to draw a triangle that has a small acute angle and then asks for another triangle with an even smaller acute angle then the student begins to realize that skinny triangles are still triangles and not sticks. Asking students for an irrational number and then asking for another forces the student to extend their example space and become more aware of the possible numbers that are available. Perhaps they are not sure if $\sqrt{7}$ is an irrational number. How will they find out? They start to check on their calculators and in the process they are recalling the conditions that are needed for an irrational number. Their appreciation of that concept has been deepened.

Asking learners for examples or non examples can also play a role in expanding example spaces. Thus, $\frac{1}{4}$ is a non example of a fraction with a repeating decimal and $\frac{2}{5}$ is a counterexample to the conjecture that fractions with non-repeating decimals have even denominators. Again, these tasks force learners to explore and extend their example spaces so that concepts are better understood. As well, experiencing extensions of example spaces can contribute to flexibility of thinking in mathematics and perhaps in any tasks involving new concepts.

In tasks 5 and 6, the teacher is trying to make students become more aware of the properties required of a quadrilateral so that in the future they have an extended notion of the concept. This pattern of questioning narrows the range of possibilities and students become more aware of the range of quadrilaterals available. Similarly, with tasks 7 and 8 involving perpendicular lines, students often assume that the segment is also bisected when in fact this is often not the case. Perhaps teachers are partly to blame for this misconception because when we draw two lines intersecting at right angles it also appears that one or both of the lines are being bisected.

Teacher Notes - Apprenticeship and Workplace Mathematics - Geometry
Outcome: Solve problems that involve isosceles and equilateral triangles, quadrilaterals, and regular polygons.

Task #1: *Draw an isosceles triangle (2 sides are the same).*

Task #2: *Draw another isosceles triangle with all of the angles different.*

Although the task is impossible, inviting the students to construct such a triangle gives them a deeper sense of the characteristics of an isosceles triangle.

Task #3: *Using your triangle from Task #1, measure the sides opposite the congruent angles. Check with your classmates and make a conclusion.*

Task #4: *Construct an equilateral triangle and make an hypothesis regarding the angle measurements. Check your measurements with a protractor and compare your findings with your classmates. Make a conclusion.*

Is it correct to say that the equilateral triangle is also isosceles? Explain.

Is it correct to say that the isosceles triangle is also equilateral? Explain.

Although we could state the properties of an isosceles triangle, inviting the students to create the triangles and exploring their properties will extend their example space and enrich their learning. The next task is a classic LGE task.(Watson and Mason, 2005)

***Task #*5**: *Draw a quadrilateral.*

Draw a quadrilateral with a pair of sides equal.

Draw a quadrilateral with a pair of sides equal and a pair of sides parallel.

Draw a quadrilateral with all of these features and a pair of opposite angles equal.

Now check that the example given at any one stage will NOT satisfy the constraint of the next stage. If necessary, make a new example that does not fit the next constraint.

This sequence of tasks challenges students to pay attention to the conditions necessary to draw a specific figure. Watson and Mason have found that many students will draw a rectangle in the first step since by convention they have associated rectangles as quadrilaterals – a case of a figural concept. These tasks could take a variety of forms such as:

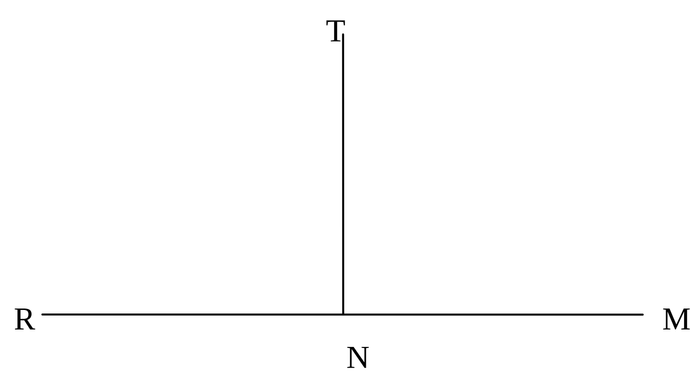
Task #6: *Draw a quadrilateral. Draw one with a right angle. Draw one with 2 right angles. Draw one with 4 right angles with diagonals that are perpendicular bisectors. Now make sure you have a different diagram at each stage and that the diagram at stage n would not fulfill the requirements at stage (n +1). So, at stage 3 your quadrilateral has 2 right angles but does not have 4 right angles with diagonals that are perpendicular bisectors. What is the name of your figure?* ________________

A sequences of tasks could be used to create whatever quadrilateral you want to focus on and one possible activity is to get students to make up their own sequence of tasks and challenge their classmates.

Task #7: *Two* **perpendicular lines** *form a pair of congruent adjacent angles. The angles will be 90 degrees. We use the symbol ⊥ to mean "is perpendicular to". Find 2 lines that are perpendicular in the classroom.*__________________________________

Think of others: ____

Lines that are perpendicular and bisect a segment are called **perpendicular bisectors.** In the



figure, TN ⊥ RM and TN bisects RM.

Task # 8: *You need a ruler to create the following examples. Pay attention to whether you need a perpendicular line, a bisector, or both.*

a. 2 segments that are perpendicular but neither one is the bisector of the other.

b. 2 segments, exactly one of which is the perpendicular bisector of the other.

c. 2 segments, each of which is the perpendicular bisector of the other.

d. 2 segments that bisect each other but are not perpendicular.

Category 3: Experiencing Structure

This is an introductory lesson to solving linear equations. Some of the students have had some experience in grade 8 solving simple linear equations and in this lesson they are expected to solve one and two step problems of various forms. Learner-generated examples are used to create the practice exercises for the class. The lesson is designed to solve linear equations with one or two unknowns but the complexity of the questions does not have to be restricted. The main intention of this lesson is to build or create equations so that students have a better understanding of how to undo or solve the equations. As students build their equations they experience the rules or methods for simplifying them.

It is important to do some modeling of what it is happening to build the equations otherwise students will have difficulty understanding what it is they are supposed to be doing. However, the aim is not to have them following algorithms that can be easily forgotten, but to help them gain an understanding of the structure of the equations. That way it is easier for them to solve questions later on. In my few experiences using this LGE lesson, I forgot to have the students solve a few of their own equations and this defeated the purpose of constructing them. It is useful to create an equation but it is essential that they also undo the equations they have built. This method of using structure to increase complexity and then using the same structure to reverse the process is known as task reversal. (Hazzan and Zazkis, 1999)

An added feature of using learner-generated examples is the ownership that is experienced by the students. Encouraging students to take responsibility for creating the practice exercises of the class seems to have a positive impact on learning and motivation. There is a sense of students taking ownership of their mathematics when they have to use their previous experiences and knowledge to create the examples. Students also enjoy trying to create problems that can be tricky or challenging for others. It seems students are more accepting of these kinds of problems rather than the ones that may appear to be imposed by mathematical authorities.

There are many other mathematical problems that involve hiding an answer in successive layers of operations that generally have to be undone to find the original solution. For example, have used it in converting quadratic equations from the vertex-graphing form of $y = a(x-p)^2 + q$ to the general form of $y = ax^2 + bx + c$. Students can create their own equation such as $y = 2(x - 4)^2 + 5$ and by expanding it they get $y = 2(x^2 – 8x + 16) + 5$ leading to $y = 2x^2 – 16x + 32 + 5$. When students start to undo their work they will recognize the first step is to factor out the 2 from $2x^2 – 16x + 32$.

Teacher Notes - Math 9 - Pattern Strand
Outcome P9.2: Model and solve situational questions using linear equations of various forms such as: $ax = b$, $\frac{x}{a} = b$, $ax + b = c$, $a(x + b) = c$, $ax + b = cx + d$.

In this lesson I am using italics to represent the voice of the teacher. *When solving equations we are really undoing or breaking down something complex into something more simple. The equation was actually constructed from a simple equation so by looking at how to build equations we may have better success solving them.* Model the first example on the board so students know what is expected of them. Although students may know basic types it is probably a good idea to start with a simple task.

Consider the equation $x = 4$. We can make it more complex by adding 3 to each side which gets us $x + 3 = 4 + 3$ or $x + 3 = 7$. How could we undo what we have built?

Students should say to subtract the 3 from both sides. The teacher can explain how addition and subtraction are opposite operations or **inverse operations.** Similarly you could do one with subtraction. If $x = 7$ and we subtract 4 then we have $x - 4 = 7 - 4$ or $x - 4 = 3$. Is your original answer still 7? Now, you can make the equations a bit more complex.

Consider the equation $x = 3$. We can make it more complex by adding 5 to each side $x + 5 = 3 + 5$ and we get $x + 5 = 8$. Now we can also add 2x to each side and get $2x + x + 5 = 8 + 2x$ so we end up with $3x + 5 = 8 + 2x$. To solve this equation you must undo what was done or do the reverse! What would you do first to undo this work? Now try to solve this equation on your student notes handout. Remember: you know the answer.

Ask a student to come to the board and solve the equation and then write $x = 5$ on the board and ask the students to create an equation for task #2 of their student notes. As well, tell them to solve the equation. Wander the room and check for misunderstandings.

Ask the students to do task #4 by creating another example and solving it. Tell them you will be using these examples as practice exercises for the whole class. Some students may introduce negative integers or division operations so the teacher will have to remind the students that **multiplication and division are inverse operations.** Tell the students to check over their examples and then to write them down on a slip of paper including their names. This information might be useful later on for clarification of their work. Write about 10 of the equations in a column on the board and have the students work in pairs or triplets at the board solving the examples. The examples that are being generated may be fairly challenging so it might be a good idea to put them down in increasing difficulty. Remember to remind students they know the answer so they need that original number.

In a small school, board-work is practical since class sizes are around 15 students. In larger schools, board-work is possible but students have to be reminded to stay on task . Students can support each other at the board and they appear to be more comfortable talking about mathematics. It seems the kinesthetic learners wake up when they get to move around a little. From the teacher perspective, board-work allows me to see who is getting it and who is having trouble. Some of the misunderstandings often provide insights into why some students are having problems. I am able to see what the student is thinking and that can be very discerning.

Lesson 3 Solving linear equations of various forms such as: ax=b, a/x= b, ax + b = c, a(x + b) = c, ax + b = cx +d

Student Notes Math 9 Pattern Strand

Outcome P9.2: Model and solve situational questions using linear equations of various forms such as ax=b, $\frac{x}{a}$ = b, ax + b = c, a(x + b) = c, ax + b = cx +d.

Task #1: We started with x = 3 and by making it more complex we finished with the equation 3x + 5 = 8 + 2x . Try solving this equation in the space provided and use the board as a guide. Remember you know the answer.

Task #2: Using the equation x = 5 make up an equation and solve it.

Task #3: Create another example and solve it. Check your work carefully.

Summary: As we solve equations, we are attempting to undo what was done to build the equation. Operations in which we do opposite operations in order to reverse the task are called **inverse operations.** So **addition/subtraction** are **inverse operations** as well as **multiplication/division.**

Category 4: Exploring Example Spaces and Experiencing Generality

A *silent lesson* uses examples generated by students as the raw material towards finding general rules or patterns. As its name implies, the teacher begins the class by writing on the board that today is a silent lesson, and then starts the lesson with an example demonstrating a certain concept or rule. It will depend on the teacher's judgment how much knowledge is given to the students. The teacher then holds up the chalk and a volunteer takes it and attempts to solve the problem, or create a new example that represents the same concept. It may be necessary for the teacher to make a correction if an example begins to mislead the students – but then again the kinds of conjectures the students suggest may enrich the learning experience.

At the beginning of a silent lesson, students will be surprised that the teacher is not talking but eventually they catch on that they need to start thinking for themselves about what the lesson involves. For some students, this may be an uncomfortable experience because they might be accustomed to relying on their classmates or the teacher to tell them the right answers. The beauty of a silent lesson is that it forces students to engage in understanding the mathematical processes involved in finding the general rule. If they can understand what is happening, they will suggest examples that fit their private conjectures. This lesson on the exponent laws is a good introductory silent lesson because students usually have success with it.

An initial reaction to this unconventional lesson is: "*That wouldn't work in my class.*" However, the silent lesson is definitely worth trying and developing as a valuable teaching strategy. It can be used with many kinds of topics, and it does not take hours of preparation time since it is the students who are supplying the examples. In my experiences with this kind of lesson, the beginning minutes of the class are awkward – similar to conversations when there are moments of silence. However, I could really sense that students were concentrating and engaged in the examples as they tried to figure out the pattern or generality. There were moments of "oh" and "ah" and because students were not allowed to talk you could see shifts of understanding and comprehension by their facial expressions. It is possible that introverted learners may enjoy this kind of lesson more than the students who tend to be verbally oriented.

Teacher Notes Foundations of Mathematics & Pre-calculus 10 Algebra and Number
Outcome: Demonstrate an understanding of powers with integral and rational exponents.

The teacher writes on the board *This is a silent lesson*. To introduce the product of powers the teacher can write down $(x^2)\cdot(x^3)=(x\cdot x)(x\cdot x\cdot x)=x^5$. Now offer the chalk to a volunteer. The volunteer will probably give a similar example and the teacher can give one more with a slight variation such as changing the variable to *y*. After another volunteer adds another example the teacher could provide one such as $5^2 \cdot 5^6 =$. The students can enter examples into their charts if they think the statements are true. It is important to provide numerous examples, because perceptions are individual and students are able to test their conjectures better if there are a variety of problems. After a few more examples, the teacher can begin the next concept.

The power of a powers concept can be taught in a similar manner but this time the teacher only puts the example on the board and not the answer: $(x^2)^3 = x^2 \cdot x^2 \cdot x^2 =$ *what?* Once again students can suggest answers and create several examples for the class. As students create examples there is a chance that the teacher may need to add examples that are challenging or may cause confusion for students. For example, questions that have a numerical base such as $(2^4)^3$ might challenge students, as could examples involving negative exponents.

Likewise a quotients of powers $\frac{x^5}{x^2}$ and the power of a product of powers $(x^4y^5)^2$ could also be taught by only writing the problem and having the students make conjectures about possible solutions. After students have created a few examples, the teacher may need to include this example: $(3x^2y)^2$ since the 3 can cause trouble. If students are writing out the repeated multiplication then there is a better chance they will understand the work and not simply just try to memorize the rule. Finally, the power of a quotient of powers can also be taught in similar fashion: $\left(\frac{x^2}{y^4}\right)^3$

After completing these tasks it may be a good idea to break the silence, summarize the key ideas, and clarify any problems with the class. Students tend to be very engaged in silent lessons because they are the ones generating the examples for the class. There seems to be an emotional connection to the lesson and a sense of relevance as they try to find the generality in the particular examples. As students create examples for the lesson they become more aware of the relationships between the variables and the exponents. Rather than being simply told the exponent rules they have been the main contributors of the examples and ideas generated in the lesson.

Here are some possible examples that teachers may expect in the chart below. It is prudent for the teacher to check that students have recorded the generalizations for each of the exponent laws in their notes. As a closure to the lesson, students could verbally state in their own words each of the laws.

Student Notes Foundations of Mathematics and Pre-calculus 10 Algebra and Number
Outcome: Demonstrate an understanding of powers with integral and rational exponents.

Use the charts below to include the examples that will be proposed on the board and summarize your findings in the space provided.

Product of Powers	Product as Repeated Multiplication	Product as a Power

Power of a Power	As Repeated Multiplication	As a Product of Factors	As a Power

Quotient of Powers	Quotient as Repeated Multiplication	Quotient as a Power

Power of a Product of Powers	As Repeated Multiplication	As a Product of Factors	As a Power

Power of a Quotient of powers	Quotient As Repeated Multiplication	As a Product of Factors	As a Quotient of Powers

Category 5: Learning A New Concept

In this lesson students are asked to discover new concepts by generating examples involving radicals and conjugacy. Several critics including Fodor (1980) have suggested that learners cannot construct a conceptually richer system than those they already know. In this lesson, we are asking learners to create examples of classes of objects they have not met before. The lesson is similar to the silent lesson except in the silent lesson the task is not as complex and students receive instant feedback if their answers are incorrect. In this task, students are searching for radicals that lead to a certain outcome and they are required to propose many examples on their quest. As well, they are required to reflect on their examples as they attempt to understand the relationships and structure of the radicals.

Students would have been familiar with simplifying radicals such as $\sqrt{24}$ and they would know that since 36 is a perfect square then $\sqrt{36}$ is 6. In grade 10 they would have learned how to multiply two binomials such as $(a + 6)(a - 4)$ but a review would be necessary. Some teachers prefer to use FOIL, but I refer to it as double distributive. In this lesson, students are asked to use binomials containing radicals of the form $(a + \sqrt{b})$ so that they can get rid of the roots and are only left with an integer. The concept to be understood is conjugacy and if the students use binomials of the form $(a + \sqrt{b})(a - \sqrt{b})$ then the end result will be an integer if a and b are integers.

Although the outcome of this lesson is conjugacy which can later be used to simplify radicals in denominators, the mathematical process of having the students try out many kinds of examples will be the focal point of using LGEs. The students will gain a deeper understanding of the nature of radicals as they reflect on the examples explored through multiplication and simplifying. A study involving a lesson somewhat similar to this was conducted by Watson and Shipman (2008) and they reported significant shifts of understanding as a result of example generation. In my classroom, I have found similar results with a few students taking about 20 minutes to reach the desired outcome. Although some students may not arrive at the conclusion that conjugate pairs are required, it is important to remember that their exploration of example spaces is a valuable experience. Throughout the lesson the teacher needs to interact with the students to clarify misunderstandings, and some students may require encouragement if they are not accustomed to "playing around" with numbers.

There are many applications of this kind of lesson using LGEs to find a special product or reach a generalization. It can be used when multiplying binomials of the form (a + b)(a – b) in which the middle linear term is eliminated and the result is $a^2 - b^2$. Students could be asked to create $30° - 60° - 90°$ triangles and look for relationships among the sides. Students can draw triangles, check the measures of the angles and reach a generality about the sum of the angles of a triangle. Some of these activities are familiar to teachers, however LGEs add the unique perspective of using examples from the students and that feature makes the learning more real, relevant and personal – and therefore more powerful.

Teacher Notes Pre-calculus 11 Algebra and Number
Outcome: Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.

In this lesson I will once again use italics to represent the voice of the teacher. Students are attempting to find the product of two special binomials containing radicals of the form

$(a + \sqrt{b})(c - \sqrt{d})$ such that the end result will not have any roots and only an integer will remain provided *a* and *b* are integers. Students have seen square roots and they have simplified roots of the form $\sqrt{36}$. They would be familiar with the product property of square roots,

$(\sqrt{a})(\sqrt{b}) = \sqrt{ab}$, as well as the multiplication of two monomials containing radicals, $(c\sqrt{a})(d\sqrt{b}) = cd\sqrt{ab}$. In grade 10, they would have multiplied binomials such as $(a+6)(a-4)$ using FOIL, double distributive, or perhaps a grid system. In any case, a revision of this process would be appropriate. One suggestion in keeping with an LGE approach is to ask the students for two binomials, write them on the board, and invite a volunteer to find the product. In this way, the teacher will have an idea of the background knowledge of the students and perhaps it may be necessary to practice a few of these binomials before tackling the ones with radicals.

Today class we are working with radicals and we want to find the product of two special binomials containing radicals of the form $(a+\sqrt{b})(c-\sqrt{d})$ *so that we no longer have any roots and we are left with only an integer. Before starting this activity, we are going to review how to multiply binomials so that you are confident multiplying binomials with radicals. Could someone give me an example of two binomials?*

After practicing several examples, the class can move on to the LGE activity. The teacher informs the class that they are to multiply two binomials of the form $(a+\sqrt{b})(c-\sqrt{d})$ so that the roots are eliminated and only an integer remains. The teacher could suggest an example such as $(5+\sqrt{3})(3+\sqrt{2})$ and walk the students through the process, asking students to provide the different terms. Now the teacher could direct the students to attempt the task of finding the two special binomials.

Your job is to find two special binomials that when multiplied will get rid of the roots and only leave an integer. You will need to change the integers and roots and you should work in pairs to help each other.

This task may require 20-30 minutes and students may not succeed in achieving conjugate pairs. However the task is still worthwhile, because they will appreciate the structure and relationships of multiplication of radicals and binomials. If students find a conjugate pair the teacher could suggest trying several more of them to make sure their generalization is correct. This is a powerful lesson because the students are learning new ideas as they generate examples and exploring example spaces in a systematic way. This lesson could be continued since the concept of conjugate pairs is needed to eliminate radicals in denominators.

One important part of this lesson is that the students share their results with the group, including the examples that did not work. In this way, the students work is still considered worthwhile and it is an opportunity for them to communicate their thoughts to the class. As well, now that the students are aware of conjugate pairs they could make up some for the class to practice. If the teacher wants to challenge them, suggestions could be made using mixed radicals such as $(5+3\sqrt{2})(5-3\sqrt{2})$.

Here are the tasks as they appear on the student pages:
*Task #1**:** Give an example showing the multiplication of two binomials.*

Task #2: Your job is to find two special binomials that when multiplied will get rid of the roots and only leave an integer. You will need to change the integers and roots and you should work in pairs to help each other. Use loose leaf to create your examples and then transfer a few onto this sheet when you are ready.

***Task #3:** Find the conjugate pair of $(4+\sqrt{2})$, and then find the remaining integer after multiplication.*

***Task #4:** Create conjugate pairs and find their product.*

***Task #5:** Create conjugate pairs containing mixed radicals and find their products.*

Summary: What did you learn?

Discussion of the Project

The Format

As teachers with busy time constraints, we tend to prefer teaching materials that require minimum preparation time and are able to engage our students. The lessons in the end product or lesson booklet contain some background information on LGEs but that knowledge has been kept to a minimum. Appendix E contains the Teacher Notes for the lessons and Appendix F has the Student Notes. Teachers seeking an opportunity to learn more about the theory of LGEs can access that information within this report in the chapter titled A Comprehensive Analysis of LGEs.

Feedback

Several grade 12 students provided some written comments expressing their personal experiences as participants of the lessons. All of the comments were positive and on a scale of 1 to 5 with 5 being the highest, students gave 3s and 4s. Students often had mixed feelings about the lessons. They liked the lessons because it forced them to think but at the same time it also caused them to be confused and they didn't like being in that place. Some comments included: *"You have to think for yourself but sometimes the lesson is confusing", "You are forced to think on your own but sometimes I don't get it",* "*When you have to figure it out, you remember it better*", "*Have to use my brain, not just told* ". On the test for this unit involving conics, students did well so one might conclude that LGEs are beneficial. In any case, they were probably not detrimental.

In a 30 minute presentation to a group of six senior math teachers at a school division in-service, teachers were receptive and interested in LGEs. The teachers generated some examples and explored some generalities using fractions. They also tried an actual lesson from this report on quadratic equations, however, they did not have to explore too much because they knew the answers right away. Most of the participants seemed generally enthused and interested in LGEs. The lessons will be accessible from a wiki site for all of the math teachers in the school division.

Personal reflections on teaching practice

LGEs take time and practice to gain proficiency but over the past several months there has been improvement in implementing them in the classroom. Now that I have a better understanding of LGEs, it is easier to introduce them spontaneously and students are starting to experience their benefits. Strangely enough, one of the side benefits of listening to the examples proposed by the students is the deeper understanding often gained by the teacher . Another benefit of LGEs is the knowledge gained when students reveal their misunderstandings in providing an incorrect example.

Limitations and implications

Although the lesson plans are a concrete product of this project and give teachers a framework to understand how LGEs can be used, for many teachers, it may require a workshop before becoming convinced of the merits of this teaching strategy. This project is the first of its kind in the province and will probably require a fair amount of practice before teachers become proficient at using LGEs. It will be exciting to be a part of the development of this concept in the ensuing months and a workshop is scheduled for the Saskatchewan Understands Math (SUM)

Conference, May 7-8 in Saskatoon. As teachers become more aware of this strategy they may want to access lessons designed in this project.

Conclusion

Mathematics teachers can use LGEs in many shapes and forms to provide quality instructional lessons. Typical of many new teaching strategies, it requires perseverance to gain familiarity with the prompts and nuances of implementing LGEs in the classroom. The lessons provided in this project are only a small sample of the infinite possibilities that LGEs offer to teachers willing to try new approaches in their daily practice. As students become accustomed to generating examples they become more engaged in the learning process. This project represents a valuable learning process for the author because it involves the creation of lesson plans of an unfamiliar concept - LGEs. In parallel fashion, our daily teaching practice can be a valuable learning process as students create their own examples in the pursuit of understanding concepts in mathematics.

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Appendix A

Researcher: Brian Crawley, Department of Curriculum Studies, College of Education
Supervisor: Dr. Janet McVittie, Department of Curriculum Studies, College of Education

Behavioural Research Ethics Board (Beh-REB)

APPLICATION FOR APPROVAL OF RESEARCH PROTOCOL

1a. **Student:** Brian Crawley, to fulfill M. Ed requirements.
1b. **Start date:** Approximately Dec.14, 2009
1c. **Completion date:** April 15, 2011

2. **Title of Study: Mathematics Lessons Using Learner-Generated Examples**

3. **Abstract:** The purpose of this project is to provide mathematics teachers an opportunity to become familiar with the implementation of Learner-generated examples (LGEs) in their daily lesson plans. The project consists of two parts: (1) organize and conduct a 1-2 hour long workshop with Math teachers describing and explaining how to implement LGEs. The format will involve the teachers participating as if they are students so they can experience the learning process of LGEs. One possible date is March 8, 2009 at the Living Sky School Division Professional Development Day. (2) develop a How-to-do-LGEs instruction manual with several detailed lesson plans that are aligned with the new WNCP Mathematics curriculum.

In the process of developing this manual, the researcher will be inviting only the grade 12 students to share their examples and journals in the analysis and production of the booklet, workshop, and presentation and perhaps for publication purposes. This invitation will occur after the students have already been assigned their final course grades.

4. **Funding:** No funding has been granted for this research. The researcher will cover all costs (photocopying and other incidentals).

5. **Expertise:** Not applicable.

6. **Conflict of Interest**: There is no conflict of interest. Research indicates that using learner-generated examples in math class provokes deeper conceptual understanding. Consequently, the students in my classes will be learning through generating their own examples. All the math students I teach will be generating examples as they fulfill their roles as students and all these students will journal about their in-class learning experiences. The students are expected to participate in this learning environment because this is the method being used by myself, the teacher. At the end of the semester, after grades have been assigned, only the grade 12 students, whom I will not teach again, will be invited to participate in the research. These students will, if they choose, allow me to use their examples and their journals for preparing the workshop for other teachers in my school division. The students will be anonymous. Their examples and their journals will not provide any information that could identify them. In this way, I will be able to discuss the examples generated by students but not reveal the source of the LGEs.

7. **Participants:** The researcher (myself) is not in a position of power relative to the participants in the study. The Grade 12 students will be asked to share their journals with me for analysis only after the semester is over. The course grades will be assigned prior to the students being recruited for the study so that I will not be in a position of power over them. The privacy and anonymity of participants will be ensured at all times.

8. **Consent:** Participation is voluntary and participants may refuse to withdraw completely without explanation or penalty and may take their data if they leave. Completion of the consent form indicates the participants' consent in the study. A parental consent form will also be used so that parents are aware of the study. See Appendix A for these consent forms.

9. **Methods/Procedures:** After the teacher has presented a lesson using LGEs, students will reply to several questions in which they can describe how well they learned the main objectives of the lesson. They will record these feelings and thoughts in their journals. These journals are part of the teaching process and will be used only for teaching, up until the semester is over. At that time, after grades have been assigned, the grade 12 students will be invited to share their journals and their learner-generated examples with me for analysis and for the booklet for teachers.

10. **Storage of Data:** All of the research data will be securely stored for a minimum of five years in a locked office at the University of Saskatchewan, as set out by Dr. Janet McVittie, Department of Curriculum Studies. After that time if the researcher decides to destroy , it will be destroyed beyond recovery**.** During the study, the data will be stored in a locked file cabinet in the researcher's home.

11. Dissemination of Results: The results of this research study are to be used in helping the researcher design better lessons using the strategy of LGEs. As well, they may be shared with other teachers at a presentation based on the implementation of this teaching tool. The data analysis may be presented at the workshop on March 8, 2010 and perhaps used in journal articles. Presentations may also be made at the College of Education (Education Curriculum 992 Master's project presentations), within the school division, or at other educational events. All identifying information of the site and the participants will be protected by the researcher and will not be used in other articles without written permission from participants.

12. Risk, Benefits, and Deception: There is no deception.

There is no potential risk to the participants in the study. The confidentiality and anonymity of the participants will be maintained at all times. At the completion of the study, participants will be informed of the impact of the study.

The potential benefit is that students will have a chance to be a part of some unique and stimulating lessons in mathematics rather than a traditional direct teaching approach. Mathematics teachers will become exposed to another teaching strategy. They will also have some unique lesson plans that may save them preparation time, motivate their students, provide an alternative approach to their teaching, and create a new interest in using constructivist teaching theories in their classrooms.

The confidentiality and anonymity of each of the participants and his or her respective school will be protected. Confidentiality of all data will be assured. No names or other means of identification will be used in any printed or published report and anonymity will be ensured through the use of pseudonyms. No comments or names that identify third parties will be included in any printed or public reports, regardless of whether participants wish to be identified.

14. Data/Transcript Release: The data will include the LGEs and the students' journals. I will make copies of these and leave the originals with the students.

15. Debriefing and feedback: Information on the impact of the workshop will be made available to the participants once the study has ended. The participants will be informed of the publication of the project and the ways of accessing it. The final project will be provided to the University of Saskatchewan.

16. Required Signatures:

________________	__________________	________________
Brian Crawley Student Researcher	Leonard Proctor, PhD ECUR Department Head	Dr. Janet McVittie Research Supervisor

17. Contact Names and Information:

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Appendix B
Letter of Consent for Students

You are invited to participate in a study entitled "*Math Lessons Using Learner-Generated Examples"*. Please read this form carefully, and feel free to ask questions you might have.

Researcher: In fulfillment of graduate studies, Brian Crawley is completing the project component of a Masters in Education degree. You can contact me at (306) 342-4600 (work)**.** I can be reached by e-mail at bk.crawley@gmail.com. My project supervisor is Dr. Janet McVittie. She can be reached at (306) 966-7582 or janet.mcvittie@usask.ca

Purpose and Procedure: This project explores a teaching strategy called learner-generated examples (LGEs) and your written feedback on the lessons may be used in the production of a set of lessons that will be available to other teachers. As well, your comments may be used in a workshop in which other teachers will be curious about your reaction to the lessons. In order to protect your interests, I am asking if you are willing to have your examples and your journals used now that you have completed the course. You are free to say no or yes to this request. Your consent will in no way affect your grades in Math and will in no way affect my attitude towards you. Although a large number of student participation will be beneficial, if everyone participated, I would have too much data, so I anticipate that some of you will say no even before I ask. You will be anonymous; your name will not be used in any quotations, and I will not tell anyone the names of those who agree to participate in the study. You may withdraw at any time during the study, and take all your data with you at that time.

Potential Risks: Because the participants for this study have been selected from a relatively small group of people, some of whom are known to each other; it is possible that you may be identifiable to other people on the basis of what you have written. I will attempt to remove identifying information and protect your anonymity in every way I can. If you are concerned at any time about anything you have written, you have full authority to withdraw your participation entirely, without explanation or penalty. I will destroy relevant material immediately and without question.

Potential Benefits: The study may aid the researcher to develop better lessons in math and may also influence math teachers to adopt this teaching strategy. However, these benefits are not necessarily guaranteed.

Storage of Data: The consent forms and written comments will be placed in sealed envelopes. These materials that are connected with this project will be safeguarded and securely stored in my research supervisor’s office at the University of Saskatchewan for a period of five years according to the University of Saskatchewan guidelines.

Confidentiality:
The data from this study may be published and presented at conferences; however, your identity will be kept anonymous unless you choose to be identified. Consent forms will be stored separately and written comments will remain anonymous. If direct quotations are used from the written comments I will assign you a pseudonym and all identifying information (such as Medstead Central School) will be removed from the report.

Right to Withdraw: Your participation is voluntary, and you may withdraw from the study for any reason, at any time, without penalty of any sort. If you quit, no one will be upset or angry and there will be no loss of grades since your mark has already been assigned to you. If you withdraw from the study at any time, any data that you have contributed will be destroyed at your request. All the information or data you provided will be deleted and destroyed.

Questions: If you have any questions concerning the study, please feel free to ask at any point; you are also free to contact the researchers at the numbers provided above if you have questions at a later time. This study has been approved on ethical grounds by the University of Saskatchewan Behavioural Research Ethics Board on January 11, 2010. Any questions regarding your rights as a participant may be addressed to that committee through the Ethics Office (966-2084). You may also contact my research supervisor Janet McVittie. She can be reached at (306) 966-7582 or janet.mcvittie@usask.ca

Follow-Up: If you are interested in learning more about results of this study please contact me and I will talk to you personally or provide you with the information that will allow you to access the final product.

Consent to Participate: I have read and understood the description provided above; I have been provided with an opportunity to ask questions and my questions have been answered satisfactorily. I consent to participate in the study described above, understanding that I may withdraw this consent at any time. A copy of this consent form has been given to me for my records.

________________________________ ________________________________

(Name of Participant) (Date)

________________________________ ________________________________

(Signature of Participant) (Signature of Researcher)

Appendix C
Parental Consent Form

Your son or daughter has been invited to participate in a study entitled "*Math Lessons Using Learner-Generated Examples"*. They are under no obligation to participate in this study. Please read this form carefully, and feel free to ask questions you might have.

Researcher: In fulfillment of graduate studies, Brian Crawley is completing the project component of a Masters in Education degree. You can contact me at (306) 342-4600 (work). I can be reached by e-mail at bk.crawley@gmail.com. My project supervisor is Dr. Janet McVittie. She can be reached at (306) 966-7582 or janet.mcvittie@usask.ca

Purpose and Procedure: This project explores a mathematics teaching strategy called learner-generated examples (LGEs) and the knowledge gained from the study may help in its implementation with other math teachers. In order to protect the interests of your son or daughter, I will adhere to the following guidelines:

1. After a lesson using LGEs, I provided your son or daughter with 3 to 4 minutes to respond on index cards to a few prompts, regarding the lesson. I have not examined the cards yet. I am asking you and your son or daughter's permission now to use those cards for my research. I will analyze the comments for trends, for commonalities and for differences.
2. I would also like to be able to use your son or daughter's LGEs for preparing a booklet for other math teachers, so they can see what these might look like and how to go about using LGEs in their classrooms.
3. Your son or daughter can change his or her mind about allowing his or her data to be used at any time up until the booklet is produced, which is expected to be in one month. If your son or daughter changes his or her mind, she or he has only to let me know, and I will remove these data from my study.

Potential Risks: Because the participants for this study have been selected from a relatively small group of people, some of whom are known to each other; it is possible that they may be identifiable to other people on the basis of what they have said. I will attempt to remove identifying information and protect your child's anonymity in every way I can.

Potential Benefits: The study may aid the researcher to develop better lessons in math and may also influence math teachers to adopt this teaching strategy. However, these benefits are not necessarily guaranteed.

Storage of Data: The consent forms and written comments will be placed in sealed envelopes. These materials that are connected with this project will be safeguarded and securely stored in my research supervisor's office at the University of Saskatchewan for a period of five years according to the University of Saskatchewan guidelines. If the researcher chooses to destroy the data, the data will be destroyed beyond recovery.

Confidentiality:
The data from this study may be published and presented at conferences; however, I have given your son or daughter the choice of having his or her identity protected. Consent forms will be stored separately and written comments will remain anonymous. If direct quotations are used from the written comments I will assign your son or daughter a pseudonym and all identifying information (for example, Medstead Central School) will be removed from the report.

Right to Withdraw: Your son or daughter's participation is voluntary, and they may withdraw from the study for any reason, at any time, without penalty of any sort. Your son or daughter may quit the research study for any reason, at any time, without explanation. If they quit, no one will be upset or angry and there will be no penalty. All the information or data they have provided will be deleted and destroyed.

Questions: If you have any questions concerning the study, please feel free to ask at any point; you are also free to contact the researchers at the numbers provided above if you have questions at a later time. This study has been approved on ethical grounds by the University of Saskatchewan Behavioural Research Ethics Board on January 11, 2010. Any questions regarding your child's rights as a participant may be addressed to that committee through the Ethics Office (966-2084). You may also contact my research supervisor Dr. Janet McVittie. She can be reached at (306) 966-7582 or janet.mcvittie@usask.ca

Follow-Up: If you are interested in learning more about results of this study please contact me and I will talk to you personally or provide you with the information that will allow you to access the final product.

Consent to Participate: I have read and understood the description provided above; I have been provided with an opportunity to ask questions and my questions have been answered satisfactorily. I consent for my son/daughter to participate in the study described above, understanding that I may withdraw this consent at any time. A copy of this consent form has been given to me for my records.

________________________________	________________________________
(Name of Participant)	(Date)
________________________________	________________________________
(Signature of Parent/Guardian)	(Signature of Researcher)

Appendix D

Questions for Written Comments

Respond to the following questions regarding this lesson.

1. What was the most important idea learned in the lesson?

2. What kinds of interesting examples were you able to generate?

3. How would you rate your understanding of the lesson on a scale of 1 to 5 with 1 being lowest and 5 being highest?

1	2	3	4	5
I don't get it	I could pass	proficient	good	I could teach this

4. What do you like about this kind of lesson?

5. What do you dislike about this kind of lesson?

Appendix E

Mathematics Lessons Using Learner-Generated Examples

The Product Component in Partial Fulfillment of the Requirements

for the Degree of Master of Education

Brian Crawley

University of Saskatchewan

A Collection of Mathematics Lessons Using Learner - Generated Examples

This compilation of lessons deals with a teaching strategy known as learner-generated examples (LGEs) in which students or learners generate examples to enhance their understanding of mathematical concepts. Although production of examples has almost always been the responsibilities of authorities there is a trend to invite learners to create the examples for the lesson. In the area of assessments, teachers are asking students to show examples as a way to demonstrate their knowledge of concepts. As well, teachers may ask students to provide examples of mathematical concepts for reasons pertaining to motivation.

In the following lessons, various uses of LGEs are described in mathematics lessons that are aligned with the new WNCP Math curriculum for grades 9 -12. These uses include (1) the role of LGEs to develop example spaces – what kinds of numbers or objects are there? (2) extending example spaces and becoming more aware of constraints (3) understanding the structure of mathematical concepts (4) exploring example spaces and experiencing patterns or generality (5) learning a new concept using LGEs as an investigative strategy.

In conventional teaching of mathematics, the teacher or the textbook are the source of the examples used in the classroom. In keeping with the philosophy of the new curriculum, these lessons invite students to be active learners so most of the examples found in the lesson are provided by the students. This is a fundamental shift for experienced teachers in which passive learning and following specific examples are the norm. Changing teaching practices is not an easy task and using LGEs (learner-generated examples) is no exception. It is uncomfortable using new teaching strategies especially when it appears one is giving up some control of the classroom, but in my experiences with LGEs, there are many benefits and insights to be gained by the teacher and student.

Each lesson plan includes the course, the strand, and the outcome. The Teacher Notes section provides the theory and rationale for using the LGEs to inform and guide the teacher. In Appendix F , you can find The Student Notes which represent a structured plan with the tasks recommended. They are a means for the student to follow the sequence of tasks and provide a written summary of the student's thinking and work. The plan is only a suggestion to be used at the discretion of the teacher. A comprehensive analysis of LGEs is available in a Final Project Report for anyone interested in learning more about this theory. As with most lesson plans, modifications and adjustments may be necessary to suit your learning environment.

Lesson 1   Rational numbers including comparing and ordering.

One of the main goals of this unit on rational numbers is for learners to have a better understanding of where numbers belong on a number line and to be more aware of the relationships between integers, fractions, decimals and mixed numbers. An easy and effective way to introduce LGEs is to simply ask students to give an example and to allow the question to be open-ended. Students may initially be uncomfortable with this kind of questioning because they may suspect the teacher has a specific concept in mind. Students may feel inhibited because perhaps their number is not good enough. As students think of examples they will start to become aware of relationships among the fractions, decimals, and place value. You may find that students need a little practice at providing examples but they will improve over time. If students think of examples that do not fit the conditions you can use the example to talk about non examples and enhance the lesson.

Teacher Notes Math 9 Number Strand
Outcome 9.2: Demonstrate an understanding of rational numbers including comparing and ordering.

Task #1: *Think of a number* ______ *Think of another* ______*And another* ______ *Share your thoughts with a classmate.*

It is important that students have the chance to share their examples in some kind of public manner such as a classroom discussion or on a blackboard. In this case, students could share with a classmate and then put their numbers on the blackboard. The next example demonstrates how to take students to a larger set of numbers if necessary.

Task #2: *Think of a rational number that lies between 1 and 2 on the number line.*

By placing constraints on the examples, students are forced to leave their example space of whole numbers and start looking for fractions or decimals.

Task#3: *Think of a fraction and then change it to a decimal.* _________ *Think of another one and change it to a decimal.* _________ *And another one* _________

The teacher may need to remind the class how to change fractions to decimals by division but most likely some grade 9 students will remember this concept. However, we know that a good teacher must be careful not to make too many assumptions. The use of LGEs allows the exploration of many kinds of fractions and corresponding decimals. The examples can be shared with the class on the board by having the students create 2 lists of terminating and repeating decimals. Now that students have created numbers in the form $\frac{\square}{\square}$, where m and n are integers and $n \neq 0$, the teacher can reinforce the term **rational number**. Note that although the teacher could investigate division by 0 , this concept requires a fair amount of time to address properly.

Task #4: *Create a fraction. ______ Create a smaller one. ______ And a smaller one. _______*

Learner – generated tasks can lead students to start making generalizations. As students make the fraction smaller they may decide to increase the size of the denominator or decrease the size of the numerator. Of course, they may change the fraction completely. Nevertheless, they may see a pattern as the denominator increases.

Task # 5: *Think of a proper fraction that has a numerator of 1. Now think of 4 more proper fractions that continue to become progressively smaller.______ ______ ______ _______ Find a proper fraction that is extremely small compared to the others. ____ What kind of pattern have you observed? ____*

Task #6: *Draw a picture that illustrates this concept.*

When students are asked to make a picture or drawing the wide range of constructions is both interesting and can generate plenty of discussion. Students tend to enjoy creating images and LGEs give students an opportunity to express their mathematical thoughts. Students like the pie diagrams or rectangular shapes but they should be encouraged to use any pictorial form.

Task #7: (a) *Think of 2 fractions that lie between $\frac{1}{5}$ and $\frac{4}{5}$. ______(b) Think of a fraction that lies between $\frac{1}{5}$ and $\frac{3}{5}$. ______(c) Think of a fraction that lies between $\frac{1}{5}$ and $\frac{2}{5}$. ______(d) Think of 3 fractions that lie between $\frac{1}{5}$ and $\frac{2}{5}$.______, ______, ______.*

Initially, students' previous knowledge will allow them to do parts a and b; however they may experience problems on the subsequent tasks. Will students be able to complete these tasks? Giving challenges to students is one of the benefits of LGEs. Students seem to enjoy this kind of exploration especially if teachers allow students to work together.

Task #8: *(a) Think of a number that is close to but less than 1. ______ (b) Now find a number that is even closer. ______ (c)Now find the largest number possible that is close to 1 using only the digits 2,5,7,and 8 ______*

The use of LGEs reinforces the concept of place value as students start to fully appreciate what kind of number is required to meet the criteria. This kind of question is much better than the typical textbook question "is 0.9 less than 1?"

Lesson 2 Solving problems that involve triangles, quadrilaterals, and regular polygons.

One of the roles of LGEs is to prompt or provoke students to engage in activities that makes them re –organize what they knew so that it fits the kind of example or object being sought. In one of these tasks, the teacher is trying to make students become more aware of the properties required of a quadrilateral so that in the future they have an extended notion of the concept. This pattern of questioning narrows the range of possibilities and students become more aware of the range of quadrilaterals available. Similarly, with the tasks involving perpendicular lines students often assume that the segment is also bisected when in fact this is often not the case. Perhaps teachers are partly to blame for this misconception because when we draw two lines intersecting at right angles it also appears that one or both of the lines are being bisected.

Teacher Notes: Apprenticeship and Workplace Mathematics Geometry Outcome: Solve problems involving triangles, quadrilaterals, and regular polygons.

Although we could state the properties of an isosceles, inviting the students to create the triangles and exploring their properties will extend their example space and enrich their learning.

Task #1: Draw an isosceles triangle (2 sides are the same).

Task #2: Draw another isosceles triangle with all of the angles different.

Although the task is impossible, inviting the students to construct such a triangle gives them a deeper sense of the characteristics of an isosceles triangle.

Task #3: Using your triangle from task #2 , measure the sides opposite the congruent angles. Check with your classmates and make a conclusion.

Task #4: Construct an equilateral triangle and make an hypothesis regarding the angle measurements. Check your measurements with a protractor and compare your findings with your classmates. Make a conclusion.

Is it correct to say that the equilateral triangle is also isosceles? Explain.

Is it correct to say that the isosceles triangle is also equilateral? Explain.

This sequence of tasks (Watson and Mason, 2005) challenges students to pay attention to the conditions necessary to draw a specific figure. Watson and Mason have found that many students will draw a rectangle in the first task since by convention they have associated rectangles as quadrilaterals- a case of a figural concept.

Task #5 Draw a quadrilateral.

Draw a quadrilateral with a pair of sides equal.

Draw a quadrilateral with a pair of sides equal and a pair of sides parallel.

Draw a quadrilateral with all of these features and a pair of opposite angles equal.

Now check that the example given at any one stage will NOT satisfy the next constraint or stage. If necessary, make a new example that does not fit the next constraint.

These tasks could take a variety of forms such as:

Task #6 Draw a quadrilateral. Draw one with a right angle. Draw one with 2 right angles. Draw one with 4 right angles with diagonals that are perpendicular bisectors. Now make sure you have a different diagram at each stage and that the diagram at stage n would not fulfill the requirements at stage (n+1). So, at stage 3 your quadrilateral has 2 right angles but does not have 4 right angles with diagonals that are perpendicular bisectors.

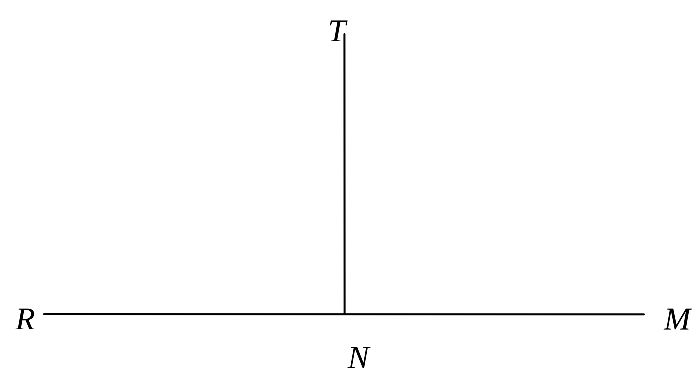
What is the name of your figure? ______________

A sequences of tasks could be used to create whatever quadrilateral you want to focus on and one possible activity is to get students to make up their own sequence of tasks and challenge their classmates.

Task #7: Two ***perpendicular lines*** *form a pair of congruent adjacent angles. The angles will be 90 degrees. We use the symbol ⊥ to mean "is perpendicular to". Find 2 lines that are perpendicular in the classroom.*___________________________

Think of others: ____

Lines that are perpendicular and bisect a segment are called **perpendicular bisectors.** In the figure, TN ⊥ RM and TN bisects RM.



Task # 8: You need a ruler to create the following examples. Pay attention to whether you need a perpendicular line , a bisector, or both.

a. 2 segments that are perpendicular but neither one is the bisector of the other.

b. 2 segments, exactly one of which is the perpendicular bisector of the other.

c. 2 segments, each of which is the perpendicular bisector of the other.

d. 2 segments that bisect each other but are not perpendicular.

Lesson 3 Solving linear equations of various forms such as: ax=b, a/x= b, ax + b = c, a(x + b) = c, ax + b = cx +d

This is an introductory lesson to solving linear equations. Some of the students have had some experience in grade 8 solving simple linear equations and in this lesson they are expected to solve one and two step problems of various forms. Learner-generated examples are used to create the practice exercises for the class. The main intention of this lesson is to build or create equations so that students have a better understanding of how to undo or solve the equations. As students build their equations, they experience the rules or methods for simplifying them. It is important to do some modeling of what it is happening to build the equations otherwise students will have difficulty understanding what it is they are supposed to be doing. It is useful to create an equation but it is essential that they also undo the equations they have built.

An added feature of using learner-generated examples is the ownership that is experienced by the students. Encouraging students to take responsibility for creating the practice exercises of the class seems to have a positive impact on learning and motivation. There is a sense of students taking ownership of their mathematics when they have to use their previous experiences and knowledge to create the examples. Students also enjoy trying to create problems that can be tricky or challenging for others. It seems students are more accepting of these kinds of problems rather than the ones that may appear to be imposed by mathematical authorities.

Teacher Notes Math 9 Pattern Strand

Outcome P9.2: Model and solve situational questions using linear equations of various forms such as: ax=b, $\frac{x}{a}$ = b, ax + b = c, a(x + b) = c, ax + b = cx +d.

In this lesson I am using italics to represent the voice of the teacher.

When solving equations we are really undoing or breaking down something complex into something more simple. The equation was actually constructed from a simple equation so by looking at how to build equations we may have better success solving them. Model the first example on the board so students know what is expected of them. Although students may know basic types it is probably a good idea to start with a simple task.

Consider the equation x = 4, we can make it more complex by adding 3 to each side which gets us x + 3 = 4 + 3 or x + 3 = 7. How could we undo what we have built?

Students should say to subtract the 3 from both sides. The teacher can explain how addition and subtraction are opposite operations or **inverse operations.** Similarly you could do one with subtraction . *If x=7 and we subtract 4 then we have x – 4 = 7 – 4 or x – 4 = 3. Is your original answer still 7?*

Now, you can make the equations a bit more complex.

Consider the equation x = 3. We can make it more complex by adding 5 to each side x + 5 = 3 + 5 and we get x + 5 = 8. Now we can also add 2x each side and get 2x + x + 5 = 8 + 2x so we end up with 3x + 5 = 8 + 2x. To solve this equation you must undo what was done or do the *reverse! What would you do first to undo this work? Now try to solve this equation on your student notes handout. Remember: you know the answer.*

Ask a student to come to the board and solve the equation and then write $x = 5$ on the board and ask the students to create an equation for task #2 of their student notes. As well, tell them to solve the equation. Wander the room and check for misunderstandings.

Ask the students to do task #4 by creating another example and solving it. Tell them you will be using these examples as practice exercises for the whole class. Some students may introduce negative integers or division operations so the teacher will have to remind the students that **multiplication and division are inverse operations.** Tell the students to check over their examples and then to write them down on a slip of paper including their names. This information might be useful later on for clarification of their work. Write about 10 of the equations in a column on the board and have the students work in pairs or triplets at the board solving the examples. The examples that are being generated may be fairly challenging so it might be a good idea to put them down in increasing difficulty. Remember to remind students they know the answer so they need that original number.

In a small school, board-work is practical since class sizes are around 15 students. In larger schools, board-work is possible but students have to be reminded to stay on task . Students can support each other at the board and they appear to be more comfortable talking about mathematics. It seems the kinesthetic learners wake up when they get to move around a little. From the teacher perspective, board-work allows me to see who is getting it and who is having trouble. Some of the misunderstandings often provide insights into why some students are having problems. I am able to see what the student is thinking and that can be very discerning.

Lesson 4 Powers with integral and rational exponents.

A *silent lesson* uses examples generated by students as the raw material towards finding general rules or patterns. As its name implies, the teacher begins the class by writing on the board that today is a silent lesson, and then starts the lesson with an example demonstrating a certain concept or rule. It will depend on the teacher's judgment how much knowledge is given to the students. The teacher then holds up the chalk and a volunteer takes it and attempts to solve the problem, or create a new example that represents the same concept. It may be necessary for the teacher to make a correction if an example begins to mislead the students – but then again the kinds of conjectures the students suggest may enrich the learning experience.

At the beginning of a silent lesson, students will be surprised that the teacher is not talking but eventually they catch on that they need to start thinking for themselves about what the lesson involves. For some students, this may be an uncomfortable experience because they might be accustomed to relying on their classmates or the teacher to tell them the right answers. The beauty of a silent lesson is that it forces students to engage in understanding the mathematical processes involved in finding the general rule. If they can understand what is happening, they will suggest examples that fit their private conjectures. This lesson on the exponent laws is a good introductory silent lesson because students usually have success with it.

An initial reaction to this unconventional lesson is: "*That wouldn't work in my class.*" However, the silent lesson is definitely worth trying and developing as a valuable teaching strategy. It can be used with many kinds of topics, and it does not take hours of preparation time since it is the students who are supplying the examples. In my experiences with this kind of lesson, the beginning minutes of the class are awkward – similar to conversations when there are moments of silence. However, I could really sense that students were concentrating and engaged in the examples as they tried to figure out the pattern or generality. There were moments of "oh" and "ah" and because students were not allowed to talk you could see shifts of understanding and comprehension by their facial expressions. It is possible that introverted learners may enjoy this kind of lesson more than the students who tend to be verbally oriented.

Teacher Notes Foundations of Mathematics and Pre-calculus 10 Algebra and Number
Outcome: Demonstrate an understanding of powers with integral and rational exponents.

The teacher writes on the board *This is a silent lesson*. To introduce the product of powers the teacher can write down $(x^2)\cdot(x^3)=(x\cdot x)(x\cdot x\cdot x)=x^5$. Now offer the chalk to a volunteer. The volunteer will probably give a similar example and the teacher can give one more with a slight variation such as changing the variable to *y*. After another volunteer adds another example the teacher could provide one such as $5^2 \cdot 5^6 =$. The students can enter examples into their charts if they think the statements are true. It is important to provide numerous examples, because perceptions are individual and students are able to test their conjectures better if there are a variety of problems. After a few more examples, the teacher can begin the next concept.

The powers-of-powers concept can be taught in a similar manner but this time the teacher only puts the example on the board and not the answer: $(x^2)^3 = x^2 \cdot x^2 \cdot x^2 =$ *what?* Once again students can suggest answers and create several examples for the class. As students create examples there is a chance that the teacher may need to add examples that are challenging or may cause confusion for students. For example, questions that have a numerical base such as $(2^4)^3$ might challenge students, as could examples involving negative exponents.

Likewise a quotients of powers $\frac{x^5}{x^2}$ and the power of a product of powers $(x^4y^5)^2$ could also be taught by only writing the problem and having the students make conjectures about possible solutions. After students have created a few examples, the teacher may need to include this example: $(3x^2y)^2$ since the 3 can cause trouble. If students are writing out the repeated multiplication then there is a better chance they will understand the work and not simply just try to memorize the rule. Finally, the power of a quotient of powers can also be taught in similar fashion: $\left(\frac{x^2}{y^4}\right)^3$

After completing these tasks it may be a good idea to break the silence, summarize the key ideas, and clarify any problems with the class. Students tend to be very engaged in silent lessons because they are the ones generating the examples for the class. There seems to be an emotional connection to the lesson and a sense of relevance as they try to find the generality in the particular examples. As students create examples for the lesson they become more aware of the relationships between the variables and the exponents. Rather than being simply told the exponent rules they have been the main contributors of the examples and ideas generated in the lesson.

Here are some possible examples that teachers may expect in the chart below. It is prudent for the teacher to check that students have recorded the generalizations for each of the exponent laws in their notes. As a closure to the lesson, students could verbally state in their own words each of the laws.

Lesson 5 Working with radicals and conjugates

In this lesson students are asked to discover new concepts by generating examples involving radicals and conjugacy. Several critics including Fodor (1980) have suggested that learners cannot construct a conceptually richer system than those they already know. In this lesson, we are asking learners to create examples of classes of objects they have not met before. The lesson is similar to the silent lesson except in the silent lesson the task is not as complex and students receive instant feedback if their answers are incorrect. In this task, students are searching for radicals that lead to a certain outcome and they are required to propose many examples on their quest. As well, they are required to reflect on their examples as they attempt to understand the relationships and structure of the radicals.

Students would have been familiar with simplifying radicals such as $\sqrt{24}$ and they would know that since 36 is a perfect square then $\sqrt{36}$ is 6. In grade 10 they would have learned how to multiply two binomials such as $(a + 6)(a - 4)$ but a review would be necessary. Some teachers prefer to use FOIL, but I refer to it as double distributive. In this lesson, students are asked to use binomials containing radicals of the form $(a + \sqrt{b})$ so that they can get rid of the roots and are only left with an integer. The concept to be understood is conjugacy and if the students use binomials of the form $(a + \sqrt{b})(a - \sqrt{b})$ then the end result will be an integer if a and b are integers.

Although the outcome of this lesson is conjugacy which can later be used to simplify radicals in denominators, the mathematical process of having the students try out many kinds of examples will be the focal point of using LGEs. The students will gain a deeper understanding of the nature of radicals as they reflect on the examples explored through multiplication and simplifying. A study involving a lesson somewhat similar to this was conducted by Watson and Shipman (2008) and they reported significant shifts of understanding as a result of example generation. In my classroom, I have found similar results with a few students taking about 20 minutes to reach the desired outcome. Although some students may not arrive at the conclusion that conjugate pairs are required, it is important to remember that their exploration of example spaces is a valuable experience. Throughout the lesson the teacher needs to interact with the students to clarify misunderstandings, and some students may require encouragement if they are not accustomed to "playing around" with numbers.

There are many applications of this kind of lesson using LGEs to find a special product or reach a generalization. It can be used when multiplying binomials of the form $(a + b)(a - b)$ in which the middle linear term is eliminated and the result is $a^2 - b^2$. Students could be asked to create $30° - 60° - 90°$ triangles and look for relationships among the sides. Students can draw triangles, check the measures of the angles and reach a generality about the sum of the angles of a triangle. Some of these activities are familiar to teachers, however LGEs add the unique perspective of using examples from the students and that feature makes the learning more real, relevant and personal – and therefore more powerful.

Teacher Notes - Pre-calculus 11 - Algebra and Number
Outcome: Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.

In this lesson I will once again use italics to represent the voice of the teacher. Students are attempting to find the product of two special binomials containing radicals of the form $(a + \sqrt{b})(c - \sqrt{d})$ such that the end result will not have any roots and only an integer will remain provided *a* and *b* are integers. Students have seen square roots and they have simplified roots of the form $\sqrt{36}$. They would be familiar with the product property of square roots, $(\sqrt{a})(\sqrt{b}) = \sqrt{ab}$, as well as the multiplication of two monomials containing radicals, $(c\sqrt{a})(d\sqrt{b}) = cd\sqrt{ab}$. In grade 10, they would have multiplied binomials such as $(a+6)(a-4)$ using FOIL, double distributive, or perhaps a grid system. In any case, a revision of this process would be appropriate. One suggestion in keeping with an LGE approach is to ask the students for two binomials, write them on the board, and invite a volunteer to find the product. In this way, the teacher will have an idea of the background knowledge of the students and perhaps it may be necessary to practice a few of these binomials before tackling the ones with radicals.

Today class we are working with radicals and we want to find the product of two special binomials containing radicals of the form $(a + \sqrt{b})(c - \sqrt{d})$ *so that we no longer have any roots and we are left with only an integer. Before starting this activity, we are going to review how to multiply binomials so that you are confident multiplying binomials with radicals. Could someone give me an example of two binomials?*

After practicing several examples, the class can move on to the LGE activity. The teacher informs the class that they are to multiply two binomials of the form $(a + \sqrt{b})(c - \sqrt{d})$ so that the roots are eliminated and only an integer remains. The teacher could suggest an example such as $(5+\sqrt{3})(3+\sqrt{2})$ and walk the students through the process, asking students to provide the different terms. Now the teacher could direct the students to attempt the task of finding the two special binomials.

Your job is to find two special binomials that when multiplied will get rid of the roots and only leave an integer. You will need to change the integers and roots and you should work in pairs to help each other.

This task may require 20-30 minutes and students may not succeed in achieving conjugate pairs. However the task is still worthwhile, because they will appreciate the structure and relationships of multiplication of radicals and binomials. If students find a conjugate pair the teacher could suggest trying several more of them to make sure their generalization is correct. This is a powerful lesson because the students are learning new ideas as they generate examples and exploring example spaces in a systematic way. This lesson could be continued since the concept of conjugate pairs is needed to eliminate radicals in denominators.

One important part of this lesson is that the teacher should make sure that students share their results with the group, including the examples that did not work. In this way, the students work is still considered worthwhile and it is an opportunity for them to communicate their thoughts to the class. As well, now that the students are aware of conjugate pairs they could make up some for the class to practice. If the teacher wants to challenge them, suggestions could be made using mixed radicals such as $(5+3\sqrt{2})(5-3\sqrt{2})$.

Lesson 6 Pythagorean Theorem

Teacher Notes: Workplace and Apprenticeship Mathematics Grade 10
Outcome: Demonstrate an understanding of the Pythagorean Theorem.

You may want to remind the students to draw their triangles carefully using a protractor or compass. In my class some students just freehanded the right angle and their numbers did not turn out too good. On the other hand, this did create some interesting discussion so maybe you do not want to remind them ahead of time.

If students have trouble thinking of examples in their lives (task #3) then perhaps suggest examples from PAA if they are doing woodworking, etc. In part E students enjoy solving their own problems. Suggest to them that they use the names of buildings or objects found in their home town.

Lesson 7 Polynomial Review

Teacher Notes: Polynomial Review Math 9

I usually give students a traditional review assignment and ask questions similar to number 16 on the student notes of Appendix F. However, in this review most of the questions involve LGEs and students did a good job of coming up with appropriate examples. In fact, the majority of students did well on this unit test and I believe getting the students to create the problems helped them understand the structure and recognize the differences between different kinds of polynomials.

Lesson 8 Analyzing Arithmetic Sequences

Teacher Notes: Grade11 Pre-calculus Algebra and Number Strand
Outcome: Analyzing arithmetic sequences

This series of tasks invites learners to create their own sequences and thus become more familiar with the structure and relationships that make up a sequence. Students will also need to find the patterns that emerge as they make up their own sequences. By putting some constraints on their sequences such as fractions and negative numbers, students are forced to extend their example spaces.

At the beginning, remember to state the main outcome of the lesson which is becoming familiar with the nature of arithmetic sequences. You could have the students share these examples orally or you could write them on the board. Remember to roam the room and check with students who may need guidance or encouragement.

In Part 2 students can explore the concept of arithmetic means and attempt to develop their own formula. Warning: For task (i) , students may not be familiar with the sequence term formula but they could still try it out.

Lesson 9 The properties of triangles to solve problems involving angles and sides

Teacher Notes - Foundations of Mathematics Outcomes - Grade 11 Geometry
Outcome: The properties of triangles to solve problems involving angles and sides.

If we provide students with the triangles in these tasks there is less understanding and fewer connections made regarding the relationships and properties. The students will gain a deeper understanding of the properties by making up their own examples. Although there are restrictions on their constructions they can still appreciate the concepts attained as they become more aware of what is possible or impossible.

Lesson 10 Graphing and writing equations of the various conic sections

Teacher Notes: Math 30C Conics

Outcome: To become aware of the various conic sections and to demonstrate skill in graphing and writing equations of the conic sections.

This lesson will not be included in the new WNCP curriculum because conics have been omitted from the new courses. The lesson is only relevant for the next 2 years. Teachers could use this lesson in segments as each conic is studied or it could be used as a review of the whole unit. The tasks asks learners to create the conics with some constraints and they try to extend the example spaces. Some of the tasks will help students become more aware of the structure of the conics as they use the given information to create the equations.

Lesson 11 Solve quadratic equations of the form $ax^2 + bx + c = 0$.

Teacher Notes Grade 11 Pre-calculus Relations and Functions

Outcome: Solve quadratic equations of the form $ax^2 + bx + c = 0$. This lesson also fits the Foundations of mathematics Grade 11 course.

Students benefit greatly when they are able to create the examples they are working because they start to appreciate the structure and complexity of the concept involved. Asking students to make up the questions is motivational and it makes them pay attention to the nuances and terminology of what is being sought. When I used this lesson I had the students work independently and then corrected each part with them. If you have not tried using LGEs this is a good starting point because the concept is fairly straight forward and students tend to do very well. The last question may need some guidance and you need to be prepared to help students with Part B (F) since decomposition or inspection is required. I usually use both but with certain students I emphasize decomposition because it gets them heading in the right direction.

Lesson 12 Using the discriminant to determine the nature of roots

Teacher Notes: Grade 11 Pre-calculus Relations and Functions
Outcome: Explain, using examples, how the discriminant may be used to determine whether a quadratic equation has two, one, or no real roots, and relate the number of zeros to the graph of the quadratic functions.

Lesson Prelude: Over the past 10 years I have taught this lesson by providing the examples of various quadratic equations in a sequential fashion so that each particular example would be representative of a different kind of root. Thus in the selection of examples, there would be one that generated two complex conjugate roots, and one with two equal rational roots, one with two unequal rational roots, and one with two unequal irrational roots. However in spite of some success with the lesson , there were always students who did not seem to understand the concept, at least for any length of time. I would point out the significance of the $\mathbf{b^2 - 4ac}$ under the radical sign but their passive role in the lesson prevented them from appreciating its importance. I recently taught this LGE lesson the last period of the day and as the students worked together at the board they were engaged and alert. In a brief discussion afterwards, they said it was much better than sitting in their desks and not really focusing. If board work is an issue, get the students to work at the board and others could write their equations at their desks.

Your goal is to get the students creating equations and solving them using the quadratic formula. Even if they do not get all of the desired roots, they will become better at using the formula since they are making more connections between the original equation and the roots that are produced. Since I have only a few students, it is easy to have all of the students at the board and their work is visible to everyone. In this way, they can see generalizations better or at least try to make suggestions as to what is happening. One surprising observation in this lesson is that students tended to make equations such as $x^2 + 2x + 5 = 0$ that produced complex numbers instead of making equations such as $x^2 + 6x + 8 = 0$ that would have produced friendlier numbers. Perhaps recent work with quadratic equations that had produced complex roots had influenced their choices of equations.

In my lesson, I asked the students to create and solve one equation and then we looked at the roots to see what was responsible for the kind of root generated. Most of the equations produced a negative value under the radical sign producing complex roots. One of the equations produced a value that was not negative, not zero and not a perfect square and the students concluded this kind of root was irrational. The students were not aware of the perfect square and zero possibilities but they did understand that the two roots were irrational and they could relate that to the $b^2 -4ac$. We discussed the implication of the part under the radical sign - the $\mathbf{b^2 -4ac}$ or **discriminant** and how it determined the nature of the roots.

I then told them there were 2 other possible roots but they had trouble finding them. After several tries, I gave them a hint that the equations were very familiar ones that could be easily factored. Some students were successful at finding an equation such as $x^2 + 7x + 10 = 0$ while one group randomly made up the equation $x^2 + 4x + 4 = 0$ that produced a zero discriminant and 2 real equal roots.

This lesson assumes that the students are familiar with the quadratic formula and complex roots. In this LGE lesson, students generate examples of quadratic equations and solve them using the quadratic formula. The nature of the roots of the equations will be analyzed by the class based on the work using the quadratic formula. Students will be asked to look for the part of the quadratic equation that is determining the nature of the roots. At the beginning of the lesson tell the students they are trying to find different kinds of roots and they are trying to figure out what it is about the formula that is determining the nature of the roots. At the end of the lesson summarize the results in the chart provided or perhaps get them to write in the missing information and then check together as a class. You can have the students use their graphic calculators to graph the equations and make conclusions as to the kind of roots and the appearance of the graphs.

Lesson 13 Factoring Trinomials

Teacher Notes - Foundation of Mathematics and Pre-calculus 10 - Algebra and Number
Outcome: Demonstrate understanding, concretely, pictorially, and symbolically, of the multiplication and factoring of polynomial expressions including trinomial factoring and relating multiplication and factoring of polynomials.

In this lesson we are factoring trinomials of the form $x^2 + bx + c = 0$. In previous work, students are aware that multiplication and factoring of polynomials are related. In Part 1 of this lesson involving learner generated examples, students are asked to give examples of a pair of integers and then to find the sum and product of the numbers. Using 2 hidden numbers they form a product and sum and challenge the class to find the numbers. In Part 2, students create binomial pairs which are multiplied and then they look for patterns in the terms to see how this can be used to factor the resulting trinomial. They are then asked to create trinomials that are factorable using integers.

For tasks #1, #2, and #3 the teacher invites students to think of 2 numbers and then to give their product and sum. The student might say 2 and 5 and the teacher can ask the class for the sum and product. Students are generally successful multiplying and adding two positive numbers but eventually the teacher can request for two negative numbers, and finally one positive and one negative number. Doing this review activity will also put students in a mindset for generating the trinomials that is required later in the lesson. Also, this task conveys to the teacher the kinds of numbers comfortable for them to multiply and add. As well, students can review the rules for multiplying integers such as positive number times a negative number results in a negative number. Some students may also have trouble with their integer rules so students can be reminded that -3 + 7 = 4 or -8 + 6 = -2.

In task #4 and #5, the students create hidden products and sums using 2 numbers. For example, they tell the class their 2 numbers have a product of 24 and a sum of 10 and the class needs to find the 2 factors that multiply to 24 and add up to get 10. This activity is the skill required to factor the trinomials later in Part 2.

In Part 2 of the lesson, students make up multiplication problems involving two binomials and they look for a pattern in the trinomial produced. Students may need a review of FOIL or double distributive but this concept should have been taught in a previous lesson. Although this lesson is often used by teachers it is important to note that there is a slight variation compared to conventional strategies. Rather than give the binomials to the students the teacher is requesting that students generate their own examples of binomials. This task may not seem difficult however students have to think more and their understanding of factoring will be enhanced when they are later asked to provide the trinomials that need to be factored.

In my experience with this lesson, students struggle in the beginning with creating trinomials that can be easily factored. They tend to go about the process in a haphazardly manner trying numbers in whatever fashion. After they create their trinomials they discover that it cannot be factored using integers. Some students may need guidance but once they catch on to the idea that any two numbers will work they are better at creating them. Again, the teacher's role is to encourage and intervene if necessary.

Lesson 14 Factoring a Difference of Squares

Teacher Notes - Foundations of Mathematics and Pre-calculus 10 - Algebra and Number
Investigating the Factoring of a Difference of Squares

Outcome: Demonstrate an understanding, concretely, pictorially, and symbolically, of the multiplication and factoring of polynomial expressions including: multiplying of monomials, binomials, and trinomials, common factors, trinomial factoring, relating multiplication and factoring of polynomials.

This lesson assumes that students are able to multiply binomials using FOIL or the double distributive method. Students are expecting a trinomial when they multiply two binomials but you are going to tell them that they are searching for a special case so that only two terms remain - a **binomial**! Rather than provide the examples, the teacher asks the students to create the binomials. Thus the teacher wants the student to understand the concept of a difference of squares by making up their own binomials and trying to find the pattern of $(x + a)(x - b)$ that results in $x^2 - a^2$. In doing so they will be able to reverse the process and be able to factor the difference of squares to get the two binomials with opposite signs.

Some students may have trouble finding the binomials but in any case students will gain a better understanding about multiplication of binomials. When you give this handout to the students make sure you do not give Part 2 because there are notes explaining the concept and students will have the answers! In Part 2, the lesson extends the concept of a difference of squares with some variation of the kinds of polynomials that they may encounter. As well, there are some non examples that do not fit the criteria such as $x^2 + 9$ which are very close to being a difference of squares but still do not qualify. We are factoring over the set of rational coefficients, Q.

Appendix F

Mathematics Lessons Using Learner-Generated Examples

The Product Component in Partial Fulfillment of the Requirements

for the Degree of Master of Education

Brian Crawley

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A Collection of Mathematics Lessons Using Learner - Generated Examples

This compilation of lessons deals with a teaching strategy known as learner-generated examples (LGEs) in which students or learners generate examples to enhance their understanding of mathematical concepts. Although production of examples has almost always been the responsibilities of authorities there is a trend to invite learners to create the examples for the lesson. In the area of assessments, teachers are asking students to show examples as a way to demonstrate their knowledge of concepts. As well, teachers may ask students to provide examples of mathematical concepts for reasons pertaining to motivation.

In the following lessons, various uses of LGEs are described in mathematics lessons that are aligned with the new WNCP Math curriculum for grades 9 -12. These uses include (1) the role of LGEs to develop example spaces – what kinds of numbers or objects are there? (2) extending example spaces and becoming more aware of constraints (3) understanding the structure of mathematical concepts (4) exploring example spaces and experiencing patterns or generality (5) learning a new concept using LGEs as an investigative strategy.

In conventional teaching of mathematics, the teacher or the textbook are the source of the examples used in the classroom. In keeping with the philosophy of the new curriculum, these lessons invite students to be active learners so most of the examples found in the lesson are provided by the students. This is a fundamental shift for experienced teachers in which passive learning and following specific examples are the norm. Changing teaching practices is not an easy task and using LGEs (learner-generated examples) is no exception. It is uncomfortable using new teaching strategies especially when it appears one is giving up some control of the classroom, but in my experiences with LGEs, there are many benefits and insights to be gained by the teacher and student.

Each lesson plan includes the course, the strand, and the outcome. The Student Notes represent a structured plan with the tasks recommended. They are a means for the student to follow the sequence of tasks and provide a written summary of the student's thinking and work. The plan is only a suggestion to be used at the discretion of the teacher. You can find the Teacher Notes section in Appendix E. They provide the theory and rationale for using the LGEs to inform and guide the teacher. A comprehensive analysis of LGEs is available in a Final Project Report for anyone interested in learning more about this theory. As with most lesson plans, modifications and adjustments may be necessary to suit your learning environment.

Lesson 1 Rational numbers including comparing and ordering.

Student Notes Math 9 Number Strand
Outcome N9.2: Demonstrate an understanding of rational numbers including comparing and ordering.

Task #1 Think of a number ______ Think of another ______And another ______ Share your thoughts with a classmate.

Task #2 Think of a rational number that lies between 1 and 2 on the number line.

Task#3 Think of a fraction and then change it to a decimal. _________ Think of another one and change it to a decimal. _________ And another one _________

Task #4 Create a fraction. ______ Create a smaller one. ______ And a smaller one. _______

Task # 5 Think of a proper fraction that has a numerator of 1. Now think of 4 more proper fractions that continue to become progressively smaller.______ ______ ______ _______ Find a proper fraction that is extremely small compared to the others. ____ What kind of pattern have you observed? ____

Task #6 Draw a picture that illustrates this concept:

Task #7 (a) Think of 2 fractions that lies between $\frac{1}{5}$ □□□ $\frac{4}{5}$. ______(b) Think of a fraction that lies between $\frac{1}{5}$ and $\frac{3}{5}$. ______(c) Think of a fraction that lies between $\frac{1}{5}$ and $\frac{2}{5}$. ______(d) Think of 3 fractions that lie between $\frac{1}{5}$ and $\frac{2}{5}$.______, ______, ______.

Task #8 (a) Think of a number that is close to but less than 1. ______ (b) Now find a number that is even closer. ______ (c)Now find the largest number possible that is close to 1 using only the digits 2,5,7,and 8 ______

Task #9 Think of a number that is close to 3 without using the number 3 or the number 9.

Assessment The intention of these lessons is not to dwell on assessment. However, it is important that teachers have a goal or plan on what it is they want to accomplish in a lesson so I have provided a short assessment for this lesson. Another advantage to LGEs is that they can provide an easy way to assess students understanding. If students can construct their own examples there is a good indication they understand the concept.

Assessment: Name ________________________________

1. Place the numbers on the number line: 4, 3.5, -1, $2\frac{1}{3}$, .6, -2.8, $\sqrt{9}$ and $\frac{3}{4}$

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2. On a small piece of paper, make up 8 rational numbers of various kinds that lie between -5 and 5 and submit them to the teacher.

3. Give an example of the following:

a. proper fraction _____ b. terminating decimal _____ c. repeating decimal ______

d. Four different kinds of rational numbers __________________________

e. Two fractions between $\frac{1}{4}$ and $\frac{2}{4}$.

f. Two decimals that are close to each other on the number line. ______________________

Lesson 2 Solving problems that involve triangles, quadrilaterals, and regular polygons.

Student Notes: Apprenticeship and Workplace Mathematics Geometry
Outcome: Solve problems involving triangles, quadrilaterals, and regular polygons.

Task #1 Draw an isosceles triangle (2 sides are the same).

Task #2 Draw another isosceles triangle one with all of the angles different.

Task #3 Using your triangle from task #2 , measure the sides opposite the congruent angles. Check with your classmates and make a conclusion.

Task #4 Construct an equilateral triangle and make an hypothesis regarding the angle measurements. Check your measurements with a protractor and compare your findings with your classmates. Make a conclusion.

Is it correct to say that the equilateral triangle is also isosceles? Explain

Is it correct to say that the isosceles triangle is also equilateral? Explain.

Task #5 Draw a quadrilateral.

Draw a quadrilateral with a pair of sides equal.

Draw a quadrilateral with a pair of sides equal and a pair of sides parallel.

Draw a quadrilateral with all of these features and a pair of opposite angles equal.

Now check that the example given at any one stage will **NOT** satisfy the next constraint or stage. If necessary, make a new example that does not fit the next constraint.

Task #6 Draw a quadrilateral.

Draw one with a right angle.

Draw one with 2 right angles.

Draw one with 4 right angles with diagonals that are perpendicular bisectors.

Now make sure you have a different diagram at each stage and that the diagram at stage *n* would not fulfill the requirements at stage *(n+1)*. So, at stage 3 your quadrilateral has 2 right angles but does not have 4 right angles with diagonals that are perpendicular bisectors.

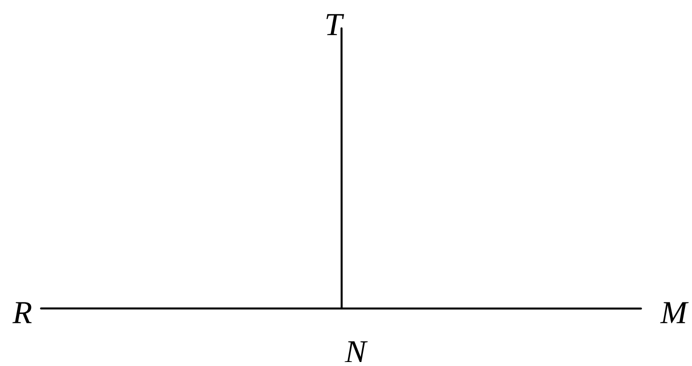
What is the name of your figure? _______________

Now make up your own set of tasks in a sequence and challenge your classmates.

Task #7 Two **perpendicular lines** form a pair of congruent adjacent angles. The angles will be 90 degrees. We use the symbol $\perp$ to mean "is perpendicular to". Find 2 lines that are perpendicular in the classroom.______________________________

Think of others: ____

In the figure, TN ⊥ RM and TN bisects RM.



Task # 8 You need a ruler to create the following examples. **Pay attention to whether you need a perpendicular line , a bisector, or both.**

a. 2 segments that are perpendicular but neither one is the bisector of the other.

b. 2 segments, exactly one of which is the perpendicular bisector of the other.

c. 2 segments, each of which is the perpendicular bisector of the other.

d. 2 segments that bisect each other but are not perpendicular.

Lesson 3 Solving linear equations of various forms such as: ax=b, a/x= b, ax + b = c,

a(x + b) = c, ax + b = cx +d

Student Notes Math 9 Pattern Strand
Outcome P9.2: Model and solve situational questions using linear equations of various forms such as ax=b, $\frac{x}{a}$ = b, ax + b = c, a(x + b) = c, ax + b = cx +d.

Task #1 We started with x = 3 and by making it more complex we finished with the equation 3x + 5 = 8 + 2x . Try solving this equation in the space provided and use the board as a guide. Remember you know the answer.

Task #2 Using the equation x = 5 make up an equation and solve it.

Task #3 Create another example and solve it. Check your work carefully.

Summary: As we solve equations, we are attempting to undo what was done to build the equation. Operations in which we do opposite operations in order to reverse the task are called **inverse operations.** So **addition/subtraction** are **inverse operations** as well as **multiplication/division.**

Lesson 4 Powers with integral and rational exponents.

Student Notes - Foundations of Mathematics and Pre-calculus 10 - Algebra and Number
Outcome: Demonstrate an understanding of powers with integral and rational exponents.

Use the charts below to include the examples that will be proposed on the board and summarize your findings in the space provided.

Product of Powers	Product as Repeated Multiplication	Product as a Power

Power of a Power	As Repeated Multiplication	As a Product of Factors	As a Power

Quotient of Powers	Quotient as Repeated Multiplication	Quotient as a Power

Power of a Product of Powers	As Repeated Multiplication	As a Product of Factors	As a Power

Power of a Quotient of powers	Quotient As Repeated Multiplication	As a Product of Factors	As a Quotient of Powers

Lesson 5 Working with radicals and conjugates

Student Notes - Pre-calculus 11 - Algebra and Number
Outcome: Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.

Task #1 Give an example showing the multiplication of two binomials.

Task #2 Your job is to find two special binomials that when multiplied will get rid of the roots and only leave an integer. You will need to change the integers and roots and you should work in pairs to help each other. Use loose leaf or the back of this paper to create your examples and then transfer a few onto this sheet when you are ready.

Task #3 Find the conjugate pair of $(4+\sqrt{2})$, and then find the remaining integer after multiplication.

Task #4 Create conjugate pairs and find their product.

Task #5 Create conjugate pairs containing mixed radicals and find their products.

Summary: What did you learn?

Lesson 6 Pythagorean Theorem

Student notes - Workplace and Apprenticeship Mathematics - Grade 10

Outcome: Demonstrate an understanding of the Pythagorean Theorem.

In a right triangle, the 2 sides that form or frame or surround the right angle are known as the **legs** of the right triangle. The side opposite (across from) the right angle is known as the **hypotenuse** of the right triangle. The hypotenuse is always the longest side in the right triangle. In the triangle on the right we name the side opposite angle A by using a single lower case letter that is the same as the corresponding vertex. So the hypotenuse is named c since it is opposite angle C. The legs are named a and b.

The Theorem of Pythagoras

Task #1 Carefully draw a right triangle using a protractor and give the triangle a name.

Ex: Δ DIG

Measure each of the legs and the hypotenuse to the nearest millimeter. Square the length of each of the legs and find the sum of these squared values. Also square the length of the hypotenuse. If you've worked carefully, the sum of the squares of the lengths of the legs should be equal to the square of the length of the hypotenuse. (well, incorrect measurements will introduce errors).

Check with your classmates and compare their results.

About 2500 years ago, Pythagoras, a Greek scholar, first proved this theorem that states: In a right triangle with hypotenuse of length c and legs of length a and b, $\mathbf{c^2 = a^2 + b^2}$

Task #2 Draw another right triangle of any size but only measure the 2 legs. Find the measure of the hypotenuse using $c^2 = a^2 + b^2$. After you get your answer use your ruler to check your result. How close were you? Why might your values be slightly off?

Task #3 Sometimes you need to find the missing leg. In triangle ABC, you

need to find x so now we get $a^2 + b^2 = c^2$ and

$x^2 + 6^2 = 11^2$

We need to subtract the 36 $x^2 = 121 - 36$

$x^2 = 85$

$x = 85$

$x = 9.2$

Can you think of examples of how this theorem would be useful in your life?

1. ____

2. ____

The Converse of the Theorem of Pythagoras

The converse of this theorem is also true. If the square of one side of a triangle is equal to the sum of the squares of the other 2 sides, then the triangle is a right triangle. Consider a triangle with lengths 3, 4 and 5. Will these sides form a right triangle? Let's check. The longest side is 5 so that is c. Using $a^2 + b^2 = c^2$ we are checking if

$3^2 + 4^2 = 5^2$.

We get $9 + 16 = 25$ Since the 2 values agree, this is a right triangle.

A. Try 8, 17, and 15. Are these the sides of a right triangle?

B. Now **make one up** that does not work but looks pretty close and show work to prove that it is not the real deal. Yes there are lots of possibilities!!

C. Challenge: **Make one up** that does work!! Hint: Think of 3, 4, and 5 and it would be kind of related, sort of like a math relative, or whatever........

D. Another common right triangle has a hypotenuse of 13. Try combinations of numbers until you find the missing legs. Hint: the sides are greater than 3.

E. Word Problem: A baseball diamond is a square that measures 90 feet on each side. How far is it from home plate to second base?

F. Your turn: Create a sweet word problem using the Pythagorean Theorem in which the missing side is one of the legs. We will solve them on the board in class so try to personalize it.

Summary: I learned ____

____

Lesson 7 Polynomial Review

Student Notes

We have been learning about polynomials and their properties. Try the following questions by yourself to refresh this knowledge and then check with a classmate to see if your examples are good. Finally put one possible answer to one of your questions on the board.

1. Give an example of a monomial. ________________________

2. Give an example of a binomial using a different variable from question 1. ___________

3. Create a trinomial that has a term that is squared, one term with degree of 1 and then a final term that has a constant. ________________________________

4. Create 2 binomials that look different but in actual fact are identical.

________________________ ________________________

5. Create a polynomial with a degree of 2 and has 3 terms including a constant term. Sketch algebra tiles to model the polynomial.

6. Create an expression that is NOT a polynomial. This is a tough one. You may need to check the definition of a polynomial found on the bottom of page 211 Math Makes Sense.

7. Create a like term for $3h^2$: ____________ Create one for 7x: __________

8. Create 2 terms that are alike: ___________ _____________

9. The following polynomial 4m + 12 represents the perimeter of a rectangle. Give some possible dimensions.

10. The sum of one polynomial is $12m^2 + 6m - 5$. One polynomial is $3m^2 - 2m + 7$. Find the other polynomial.

11. Create a polynomial that is added to $5x^2 – 7x + 1$ to get a sum of $–x^2 + 3x – 8$.

12. Create a polynomial subtraction question. Answer it .

13. (a) Find the product of $3(2a^2 - 5a + 4)$. ______________________________
(b) Now make up a question so that a constant is multiplying a binomial.

14. Find the quotient of $\dfrac{8b^2 + 12b - 6}{-2}$

15. Make up a quotient question in which a polynomial is divided by a constant.

16. Multiply or divide.

a. $3m(2m - 4)$ ____________ b. $-7x(2x + 3)$ ________________

c. $(-2y - 5)(-y)$ ____________ d. $(3h)(-5h)$ _________________

17. Find the quotient.

a. $\dfrac{10x^2 + 12x}{2x}$

b. $\dfrac{30r^2 + 20r}{-5r}$

18. The polynomial $16x^2 + 8x$ can be represented by the areas of rectangles with different dimensions. Sketch and label the dimension for one such rectangle. Write a division statement for your rectangle. Can you think of others?

Lesson 8 Analyzing Arithmetic Sequences

Student Notes Grade 11 Pre-calculus Pathway Sequences and Series

A. Think of a sequence of 3 numbers that differ by 2 : ____________________________

B. Now think of another one with 3 numbers that differ by 2 ___________________________________

C. Now think of another one with 3 numbers that differ by 2:

____

The above sequences are called **arithmetic sequences** because all of the terms will differ from one another by the same amount. In the above sequences each term is differing by 2 so we say the **common difference or d** is 2.

D. Create another example of an arithmetic sequence that differs by 4.

____

E. Make up a sequence that appears to be arithmetic but in fact is not.

____

F. Consider these sequences: 3, 5, 7, 9 ...... or 3, 8, 13, 18......

a. How are they the same? _______________________________________

b. What is different about them? __________________________________

c. What are the next 2 terms of each sequence? ____ ____ and ___ _____

G. Make up an arithmetic sequence that does not start with a 3 and show it to your partner. Ask them to state the first term or **a** and the common difference or **d.**

____

H. Make up another one involving fractions – hmmm, a little more challenging.

____

I. Construct a sequence that is decreasing without negative numbers.

____

What do you notice about the common difference? _______________

J. Construct a sequence that is decreasing using negative numbers.

____

K. Write a formula for finding the common difference or **d**.

____

Part 2 Arithmetic Means

Consider the sequence 2, 5, 8. The sequence has 3 terms and the middle term, in this case 5, has a special name. It is called an **arithmetic mean.**

A. What is the arithmetic mean for the sequence 5, 10, and 15? _______

B. How about this sequence of 4, ____, 8.

C. What about 10, ______30? _______ How are you getting this number?

D. Can you write a formula to find the arithmetic mean?

E. Make up an example to test it and be sure to check with your partner.

F. Consider the sequence 3, _____, _____, _____, 19. There are 3 arithmetic means between 3 and 19. What are the missing arithmetic means? Use your formula.

G. Make up an arithmetic sequence with 3 means between 2 given numbers. ________________________________

H. Now try to find the 2 means between 6 and 18. Describe your attempts.

____

I. How can you solve this one? Hint: use the formula

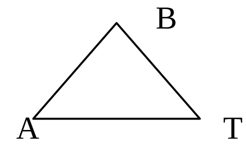
$t_n = a + (n - 1)d$

J. Try this one 2, _____, _____, _____, _____ 47.

Lesson 9   The properties of triangles to solve problems involving angles and sides

Student notes - Foundations of Mathematics Outcomes - Grade 11 Geometry
Outcome: The properties of triangles to solve problems involving angles and sides.

A **triangle** is formed by connecting 3 non- collinear points with line segments. The points are the vertices and the lines are the sides. Use the symbol Δ to represent triangle. For ΔBAT, name the 3 vertices _____, _____, ______.



Classifying triangles according to **side length**

Using your ruler and protractor carefully construct the following triangles and compare your results with your classmates.

1. Create a triangle with all sides different:
This kind of triangle is called **scalene**.

2. Create a triangle with 2 sides the same length:
This is called an **isosceles** triangle. Find the measurements of the angles. _____, ______, _____ .Check with your classmates and draw a conclusion regarding the angles.

3. Now draw a triangle with 3 sides congruent .This is called an **equilateral** triangle. Using your protractor, measure the angles. _________ Were you expecting this result?

Classifying triangles according to angle size
4. Construct a triangle with all 3 angles acute. This is called an **acute triangle.**

5. Make an **obtuse triangle** by having one of the angles obtuse (greater than 90°).

6. Create a **right triangle** by having one of the angles 90 degrees.

7. Draw a triangle with all of the angles the same size. This is called an **equiangular triangle.**

The Five Triangle Properties

Property 1: Check the sum of the three angles in any 2 of your triangles. What do you get?

____

Compare your values with those of your classmates. What can you conclude?

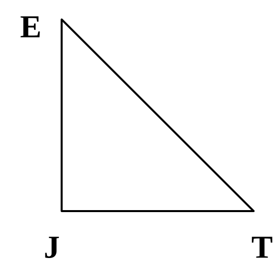
____

Property 2: If 2 angles of one triangle are equal to two angles of a second triangle, then the remaining angles in each triangle are also equal. Draw two triangles with two angles congruent. What do you know about the other angles? ____

Property 3: Since the 3 angles of an equiangular triangle are equal, and the 3 angles of a triangle add up to 180 degrees, the angles must be _________ degrees.
Check these measurements on the triangle from question 7.

Property 4: Try to draw a triangle with 2 obtuse angles.
What is the problem? Now try to draw a triangle with 2 right angles. What is going on? Conclusion: A triangle cannot have ________________________NOR ___________________
____

Property 5: Measure the acute angles of the right triangle JET.
What do they add up to? _____________These angles are complementary. How could we prove this?______________
____



Lesson 10 Graphing and writing equations of the various conic sections

Student Notes Math 30C Conics Review

1. Construct the equation of a circle that fits entirely inside quadrant lV.

2.Create an equation of a circle in general form that can be easily changed to standard form so that there are only integers in the center and radius.

3. Construct the equation of a parabola with the required conditions:

a. The vertex is (5,-3) , it opens left , and p = -4

b. The vertex is (-2, 7) and the focus is (-2, 10)

4. Find the equations of a horizontal ellipse and a vertical hyperbola such that there are 4 points of intersection. They can both have centers at the origin

5. Create an example of the following conics.

(a) Line - ____________________ ____________________

(b) Circle ____________________ ____________________

(c) Parabola (H)____________________ (V)____________________

(d) Ellipse (H) ____________________ (V) ____________________

(e) Hyperbola (H) ____________________ (V) ____________________

6. Create an example of an ellipse that is almost close to being a circle.

7. Create the equations of 2 ellipses such that one is vertical and located in the fourth quadrant and the other is horizontal in the first quadrant. As well, the ellipses should **intersect** in **2** places.

8. Make up 5 equations consisting of one circle, one parabola, a line, an ellipse and a hyperbola. Mix up the order and then get a classmate to try to identify the conics.

1.

2.

3.

4.

5.

Lesson 11 Solve quadratic equations of the form $ax^2 + bx + c = 0$.

Student notes - Grade 11 Pre-calculus - Relations and Functions

Outcome: Solve quadratic equations of the form $ax^2 + bx + c = 0$. This lesson also fits the Foundations of Mathematics Grade 11 course.

A. You have seen quadratic equations before but you may need some brushing up on this concept. Can you think of one? ________________________

B. Can you think of another one? ________________________________

C. What do these equations have in common? _____________________

D. Can you think of a number that solves the linear equation $x + 3 = 0$ ____
How do you know this number is correct? _______________________

E. Think of any number _____ and multiply it by 0. What is the product?
____

F. Try another number. ___________ And another _________________

G. Now look at this problem in which 2 binomials are multiplied.
$(x + 4)(x - 5) = 0$. Try to think of numbers that will make this equation true.
Hint: There are 2 of them. ________and ________.

H. Check if your answers are correct by substitution.

I. Make up another similar problem and check your solution set.

J. Make up an equation in which the solutions are 5 and -7.

K. Make up an equation that has solutions of -7 and 7.

State your conclusion regarding this principle:

The principle you are attempting to write about is known as the **zero product principle**. It can be written as: **If AB = 0 then A= 0 or B = 0.**

Part B

Sometimes we are not given 2 binomials in factored form. We might have to solve a quadratic equation like this $x^2 - x - 20 = 0$.

A. What might be a method to solving this equation? _________________
Try that: _________________________
The solution set or s.s. would be { _____, _____ }.

B. Make up a quadratic equation that is easily factored with integers as solutions and find the solution set. ____
Hint: if you are having trouble try starting with any 2 binomials.

C. Make one up for your classmate and ask them to find the solution set. Keep the numbers reasonable.

____

D. Consider this equation $x^2 = 7x + 18$. How is it different than the above examples? ____
What needs to be done to make it similar? _________________________

E. Try to solve it.

F. How is this equation different from the one above? $8x^2 + 14x - 15 = 0$.
____

G. How can it be factored? ________________________________

Try it. If you need help, see me.

Part C

A. The solutions for a quadratic equation are 6 and -6. Can you find the 2 binomials? ___________ and _____________
What is the resulting equation ? ______________________________.
Why is this interesting?_____________________________________

B. You have seen these polynomials before; they are called __________________________. Solve this one: $x^2 - 4 = 0$
____

C. Construct one of these equations and ask your classmate to solve it.

___

D. Here is another one but slightly concealed; $3x^2 - 9x = 0$.

What strategy could you use to solve it? ______________________________

Try solving it.

E. Try to make up a similar example and ask me to solve it.

___

G. Challenge: $\frac{x}{x-4} + \frac{x-2}{x-5} = 4$

Lesson 12 Using the discriminant to determine the nature of roots

Student Notes Grade 11 Pre-calculus Relations and Functions

Outcome: Explain, using examples, how the discriminant may be used to determine whether a quadratic equation has two, one, or no real roots, and relate the number of zeros to the graph of the quadratic functions.

Task #1 You are learning about the different kinds of roots that can be formed using the quadratic formula. Create an example of a quadratic equation and then solve it using the quadratic formula. After solving the equation observe the solutions of other equations and try to form a relationship between the nature of the roots and work involved using the formula.

Task #2 There are 4 kinds of roots that can be generated. Make up more examples of quadratic equation and compare your solutions with your classmates. Summarize the results in the chart provided.

The Discriminant **$D = b^2 - 4ac$**	The Nature of the Roots of the Quadratic Equation $ax^2 + bx + c = 0$	Sketch the graph
Negative		
Zero		
Positive - a perfect square		
Positive - Not a perfect square		

Task #3 Practice: Using your chart above, describe the roots of the quadratic equations by finding the discriminant.

a. $x^2 + 4x - 5 = 0$

b. $-3b^2 + 4b + 1 = 0$

c. $x^2 + 2x = -7$

d. $3x^2 - 4x + \frac{4}{3} = 0$

Lesson 13 Factoring Trinomials

Student notes - Foundation of Mathematics and Pre-calculus 10 - Algebra and Number Outcome: Demonstrate an understanding, concretely, pictorially, and symbolically, of the multiplication and factoring of polynomial expressions including trinomial factoring.

Part 1

Task #1 Think of any 2 positive numbers. What is the sum and product? Think of two more positive numbers. Find their sum and product.

Task #2 Think of any 2 negative numbers. Find their sum and product.

Task #3 Think of a positive and negative number. Find their sum and product.

Task #4 Think of 2 positive numbers such as 4 and 6. The product is 24 and the sum is 10. You need to create a sum and product from 2 numbers and then share your numbers with the class. The class will try to determine the 2 numbers from which the sum and products were derived. In the above example, you would say "the product is 24 and the sum is 10. The class would respond with "the numbers are 4 and 6".

Task #5 Think of any products and sums but this time include at least one negative number. For example, a product of -12 and a sum of -1, the numbers would be -4 and 3.

Part 2

Task #1 You have multiplied binomials of the form (x + 4) (x + 5) which resulted in the trinomial $x^2 + 9x + 20$. Complete the chart with pairs of binomials of the form (x + a)(x + b) and find their products.

Binomial	Use FOIL, Double Distributive, or Mental Math	Trinomial
(x + 4)(x + 5)	$x^2 + 5x + 4x + 20$	$x^2 + 9x + 20$

Summary: Can you see a pattern relating the trinomial formed and the original binomials? How would you factor the trinomial $x^2 + 9x + 20$.

Now use the chart to make up trinomials that can be easily factored. The first one is done for you. Follow the pattern of using positive terms in the trinomial so the sign pattern is + + +.

Trinomial	Factors
$x^2 + 9x + 20$	$(x + 5)(x + 4)$

Task #2 You have multiplied binomials of the form (x - 2) (x - 7) which resulted in the trinomial $x^2 - 9x + 17$. Complete the chart with pairs of binomials of the form (x -a)(x - b) and find their products.

Binomial	Use FOIL, Double Distributive, or Mental Math	Trinomial
$(x - 2)(x - 7)$	$x^2 - 2x - 7x + 14$	$x^2 - 9x + 14$

Summary: Can you see a pattern relating the trinomial formed and the original binomials? How would you factor the trinomial $x^2 - 9x + 14$?

Now use the chart to make up trinomials that can be easily factored. The first one is done for you. Follow the pattern of using positive terms in the trinomial so the sign pattern is + - +.

Trinomial	Factors
$x^2 - 9x + 14$	(x - 2) (x - 7)

Task #3 Multiplying two binomials of the form (x + 3)(x - 6) results in the trinomial $x^2 -3x -18$. Complete the chart with pairs of binomials of the form (x - a)(x + b) and find their products.

Binomial	Use FOIL, Double Distributive, or Mental Math	Trinomial
(x + 3)(x - 6)	$x^2 - 6x + 3x - 18$	$x^2 -3x -18$.

Summary: Can you see a pattern relating the trinomial formed and the original binomials? How would you factor the trinomial $x^2 - 3x – 18$?

Now use the chart to make up trinomials that can be easily factored. The first one is done for you. Follow the pattern of using positive terms in the trinomial so the sign pattern is
+ - - or + + - .

Trinomial	Factors
x^2 -3x -18.	(x + 3)(x - 6)

Task #4 Now create a variety of trinomials using the different combinations of sign patterns that are possible.

Include a few that are prime. We will put some on the board and factor them.

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

5. ______________________________

6. ______________________________

Lesson 14 Factoring a Difference of Squares

Student Notes Foundations of mathematics and Pre-calculus 10 Algebra and Number

Outcome: Demonstrate understanding, concretely, pictorially, and symbolically, of the multiplication and factoring of polynomial expressions including: multiplying of monomials, binomials, and trinomials, common factors, trinomial factoring, relating multiplication and factoring of polynomials.

Recall that a binomial has 2 terms so one example could be x + 3. Think of another that uses the variable x. ______________. Now think of other binomials that use a different variable like *y*. So for example we have 3y – 4. ______________________ _______________________.

When we multiply 2 binomials we usually get a trinomial. Using double distributive and multiplying

(x + 4) (x +5) we multiply the first x onto the second x _______

multiply the first x onto the 5 _______

multiply the 4 onto the x _______

multiply the 4 onto the 5 _______

We combine the middle terms and get the trinomial $x^2 + 9x + 20$.

Likewise we can multiply (x – 2)(x + 6) and get ________ ________ _______ _______

which becomes the trinomial $x^2 + 4x - 12$.

Task #1 Will we always get a trinomial?? Your job is to create binomials so that when you multiply them you end up with a **BINOMIAL**! Use the space below or the back of this paper to explore. After you find 2 binomials that work, try a few more to make sure it works.

Task #2 Then write down the conditions for this special binomial to occur. These are your **study notes**! When you are finished tell me and I will give you **Part 2** of this lesson. Please do not tell your classmates otherwise you are interfering with their learning process.

Part 2

The special binomial you discovered is called a **difference of squares.** When we multiply 4 x 6 and get the product 24 , the 4 and the 6 are called **factors**. Consider the product of the binomials (x + 4) (x – 4). We get $x^2 – 16$. The (x+4) and (x-4) are called the factors so when we meet a difference of squares such as $x^2 -16$ and we wish to find the factors think of the 2 binomials that when multiplied get this result. We are **reversing the multiplication process! Pay attention to the sign pattern.**

Practice : 1.Factor $x^2 – 25$ = ________________________

2. Factor $x^2 – 64$ = ______________________

3. In the above examples the degree of the binomials is 2. Remember we look at the largest exponent of each term. Now try to create perfect squares with a degree of 2 using different variables such as x,y, etc.

__________________________ ______________________ ________________________

4. Try to make up examples that have a degree of **4** or **6** and factor them. If you are not sure of your answers how can you check your work?

A. ___________________

B.___________________

C. _____________________

5. Make up an example that has a coefficient such as $49b^2 – 9w^4$. ______________________

6. Can you factor $x^2 – 11$? ________________________________

7. How about $x^2 + 4$?(over Q) ____

8. Make one up that cannot be factored as a difference of squares. ___________________

9. **Tricky stuff!!** The binomial $3a^3 – 12a$ does not appear to be a difference of squares **however** we can find a **common factor that can be removed.** Now you can see the difference of squares. Try it.

9. Challenging one: $(x+ y)^2 – 81$