Real-life data can be modeled by several types of equations. A rational equation can be used to model the voting-age population of the United States.

SELECTED APPLICATIONS

Equations and inequalities have many real-life applications. The applications listed below represent a small sample of the applications in this chapter.

- Population Statistics, Exercise 67, page 88
- Anthropology, Exercise 95, page 96
- Radio Waves, Exercise 61, page 107
- Data Analysis: Movie Tickets, Exercise 119, page 123
- Money in Circulation, Exercise 131, page 124
- Voting Population, Exercise 94, page 142
- Data Analysis: IQ Scores and GPA, Exercise 99, page 152
- Height of a Projectile, Exercise 67, page 162
The Graph of an Equation

In Section P.7, you used a coordinate system to represent graphically the relationship between two quantities. There, the graphical picture consisted of a collection of points in a coordinate plane.

Frequently, a relationship between two quantities is expressed as an equation in two variables. For instance, is an equation in and . An ordered pair is a solution or solution point of an equation in and if the equation is true when is substituted for and is substituted for . For instance, is a solution of because is a true statement.

In this section you will review some basic procedures for sketching the graph of an equation in two variables. The graph of an equation is the set of all points that are solutions of the equation.

Determining Solutions

Determine whether (a) and (b) are solutions of the equation 

\[ y = 10x - 7 \]

Solution

a. \[ y = 10x - 7 \]
   \[ 13 = 10(2) - 7 \]
   \[ 13 = 13 \]
   Because the substitution does satisfy the original equation, you can conclude that the ordered pair \((2, 13)\) is a solution of the original equation.

b. \[ y = 10x - 7 \]
   \[ -3 \neq 10(-1) - 7 \]
   \[ -3 \neq -17 \]
   Because the substitution does not satisfy the original equation, you can conclude that the ordered pair \((-1, -3)\) is not a solution of the original equation.

The basic technique used for sketching the graph of an equation is the point-plotting method.

Sketching the Graph of an Equation by Point Plotting

1. If possible, rewrite the equation so that one of the variables is isolated on one side of the equation.
2. Make a table of values showing several solution points.
3. Plot these points on a rectangular coordinate system.
4. Connect the points with a smooth curve or line.
When making a table of solution points, be sure to use positive, zero, and negative values of $x$.

**Example 2**  Sketching the Graph of an Equation

Sketch the graph of $y = 7 - 3x$.

**Solution**

Because the equation is already solved for $y$, construct a table of values that consists of several solution points of the equation. For instance, when $x = -1$,

\[ y = 7 - 3(-1) = 10 \]

which implies that $(-1, 10)$ is a solution point of the graph.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = 7 - 3x$</th>
<th>$(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>10</td>
<td>(-1, 10)</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
<td>(0, 7)</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>(1, 4)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(2, 1)</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
<td>(3, -2)</td>
</tr>
<tr>
<td>4</td>
<td>-5</td>
<td>(4, -5)</td>
</tr>
</tbody>
</table>

From the table, it follows that

$$(-1, 10), (0, 7), (1, 4), (2, 1), (3, -2), \text{ and } (4, -5)$$

are solution points of the equation. After plotting these points, you can see that they appear to lie on a line, as shown in Figure 1.1. The graph of the equation is the line that passes through the six plotted points.

![Figure 1.1](image-url)

**CHECKPOINT** Now try Exercise 5.
Example 3 Sketching the Graph of an Equation

Sketch the graph of

\[ y = x^2 - 2. \]

**Solution**

Because the equation is already solved for \( y \), begin by constructing a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = x^2 - 2 )</td>
<td>(2)</td>
<td>(-1)</td>
<td>(-2)</td>
<td>(-1)</td>
<td>(2)</td>
<td>(7)</td>
</tr>
<tr>
<td>((x, y))</td>
<td>((-2, 2))</td>
<td>((-1, -1))</td>
<td>((0, -2))</td>
<td>((1, -1))</td>
<td>((2, 2))</td>
<td>((3, 7))</td>
</tr>
</tbody>
</table>

Next, plot the points given in the table, as shown in Figure 1.2. Finally, connect the points with a smooth curve, as shown in Figure 1.3.

One of your goals in this course is to learn to classify the basic shape of a graph from its equation. For instance, you will learn that the linear equation in Example 2 has the form

\[ y = mx + b \]

and its graph is a line. Similarly, the quadratic equation in Example 3 has the form

\[ y = ax^2 + bx + c \]

and its graph is a parabola.

The point-plotting method demonstrated in Examples 2 and 3 is easy to use, but it has some shortcomings. With too few solution points, you can misrepresent the graph of an equation. For instance, if only the four points

\[-2, 2), (1, -1), (1, -1),\) \text{ and } (2, 2) \]

in Figure 1.2 were plotted, any one of the three graphs in Figure 1.4 would be reasonable.

The point-plotting method demonstrated in Examples 2 and 3 is easy to use, but it has some shortcomings. With too few solution points, you can misrepresent the graph of an equation. For instance, if only the four points

\[-2, 2), (1, -1), (1, -1),\) \text{ and } (2, 2) \]

in Figure 1.2 were plotted, any one of the three graphs in Figure 1.4 would be reasonable.
Intercepts of a Graph

It is often easy to determine the solution points that have zero as either the \( x \)-coordinate or the \( y \)-coordinate. These points are called \textit{intercepts} because they are the points at which the graph intersects or touches the \( x \)- or \( y \)-axis. It is possible for a graph to have no intercepts, one intercept, or several intercepts, as shown in Figure 1.5.

Note that an \( x \)-intercept can be written as the ordered pair \((x, 0)\) and a \( y \)-intercept can be written as the ordered pair \((0, y)\). Some texts denote the \( x \)-intercept as the \( x \)-coordinate of the point \((a, 0)\) [and the \( y \)-intercept as the \( y \)-coordinate of the point \((0, b)\)] rather than the point itself. Unless it is necessary to make a distinction, we will use the term \textit{intercept} to mean either the point or the coordinate.

Example 4 \hspace{1em} \textbf{Identifying \( x \)- and \( y \)-Intercepts}

Identify the \( x \)- and \( y \)-intercepts of the graph of
\[
y = x^3 + 1
\]
shown in Figure 1.6.

Solution

From the figure, you can see that the graph of the equation \( y = x^3 + 1 \) has an \( x \)-intercept (where \( y \) is zero) at \((-1, 0)\) and a \( y \)-intercept (where \( x \) is zero) at \((0, 1)\).

\hspace{1em} \hspace{1em} \hspace{1em} Now try Exercise 9.
Symmetry

Graphs of equations can have symmetry with respect to one of the coordinate axes or with respect to the origin. Symmetry with respect to the \(x\)-axis means that if the Cartesian plane were folded along the \(x\)-axis, the portion of the graph above the \(x\)-axis would coincide with the portion below the \(x\)-axis. Symmetry with respect to the \(y\)-axis or the origin can be described in a similar manner, as shown in Figure 1.7.

**Graphical Tests for Symmetry**

1. A graph is symmetric with respect to the \(x\)-axis if, whenever \((x, y)\) is on the graph, \((x, -y)\) is also on the graph.
2. A graph is symmetric with respect to the \(y\)-axis if, whenever \((x, y)\) is on the graph, \((-x, y)\) is also on the graph.
3. A graph is symmetric with respect to the origin if, whenever \((x, y)\) is on the graph, \((-x, -y)\) is also on the graph.

**Example 5** Testing for Symmetry

The graph of \(y = x^2 - 2\) is symmetric with respect to the \(y\)-axis because the point \((-x, y)\) is also on the graph of \(y = x^2 - 2\). (See Figure 1.8.) The table below confirms that the graph is symmetric with respect to the \(y\)-axis.

<table>
<thead>
<tr>
<th></th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(y)</td>
<td>7</td>
<td>2</td>
<td>(-1)</td>
<td>(-1)</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>((x, y))</td>
<td>((-3, 7))</td>
<td>((-2, 2))</td>
<td>((-1, -1))</td>
<td>((1, -1))</td>
<td>((2, 2))</td>
<td>((3, 7))</td>
</tr>
</tbody>
</table>

Now try Exercise 15.
Using Symmetry as a Sketching Aid

Solution

Of the three tests for symmetry, the only one that is satisfied is the test for -axis symmetry because is equivalent to . So, the graph is symmetric with respect to the -axis. Using symmetry, you only need to find the solution points above the -axis and then reflect them to obtain the graph, as shown in Figure 1.9.

Now try Exercise 29.

Sketching the Graph of an Equation

Solution

This equation fails all three tests for symmetry and consequently its graph is not symmetric with respect to either axis or to the origin. The absolute value sign indicates that is always nonnegative. Create a table of values and plot the points as shown in Figure 1.10. From the table, you can see that when . So, the -intercept is (0, 1). Similarly, when . So, the -intercept is (1, 0).

Now try Exercise 33.
Throughout this course, you will learn to recognize several types of graphs from their equations. For instance, you will learn to recognize that the graph of a second-degree equation of the form

$$y = ax^2 + bx + c$$

is a parabola (see Example 3). The graph of a circle is also easy to recognize.

**Circles**

Consider the circle shown in Figure 1.11. A point \((x, y)\) is on the circle if and only if its distance from the center \((h, k)\) is \(r\). By the Distance Formula,

$$\sqrt{(x - h)^2 + (y - k)^2} = r.$$  

By squaring each side of this equation, you obtain the **standard form of the equation of a circle**.

**Standard Form of the Equation of a Circle**

The point \((x, y)\) lies on the circle of radius \(r\) and center \((h, k)\) if and only if

$$(x - h)^2 + (y - k)^2 = r^2.$$  

From this result, you can see that the standard form of the equation of a circle with its center at the origin, \((h, k) = (0, 0)\), is simply

$$x^2 + y^2 = r^2.$$  

Circle with center at origin

**Example 8** Finding the Equation of a Circle

The point \((3, 4)\) lies on a circle whose center is at \((-1, 2)\), as shown in Figure 1.12. Write the standard form of the equation of this circle.

**Solution**

The radius of the circle is the distance between \((-1, 2)\) and \((3, 4)\).

\[
\begin{align*}
r & = \sqrt{(x - h)^2 + (y - k)^2} \\
& = \sqrt{(3 - (-1))^2 + (4 - 2)^2} \\
& = \sqrt{4^2 + 2^2} \\
& = \sqrt{16 + 4} \\
& = \sqrt{20}
\end{align*}
\]

Using \((h, k) = (-1, 2)\) and \(r = \sqrt{20}\), the equation of the circle is

\[
(x - h)^2 + (y - k)^2 = r^2
\]

$$[x - (-1)]^2 + (y - 2)^2 = \left(\sqrt{20}\right)^2$$

$$[x + 1]^2 + (y - 2)^2 = 20.$$  

Standard form

Now try Exercise 53.

You will learn more about writing equations of circles in Section 4.4.
Application

In this course, you will learn that there are many ways to approach a problem. Three common approaches are illustrated in Example 9.

A Numerical Approach: Construct and use a table.

A Graphical Approach: Draw and use a graph.

An Algebraic Approach: Use the rules of algebra.

Example 9  Recommended Weight

The median recommended weight \( y \) (in pounds) for men of medium frame who are 25 to 59 years old can be approximated by the mathematical model

\[
y = 0.073x^2 - 6.99x + 289.0, \quad 62 \leq x \leq 76
\]

where \( x \) is the man’s height (in inches). (Source: Metropolitan Life Insurance Company)

a. Construct a table of values that shows the median recommended weights for men with heights of 62, 64, 66, 68, 70, 72, 74, and 76 inches.

b. Use the table of values to sketch a graph of the model. Then use the graph to estimate graphically the median recommended weight for a man whose height is 71 inches.

c. Use the model to confirm algebraically the estimate you found in part (b).

Solution

a. You can use a calculator to complete the table, as shown at the left.

b. The table of values can be used to sketch the graph of the equation, as shown in Figure 1.13. From the graph, you can estimate that a height of 71 inches corresponds to a weight of about 161 pounds.

\[
\begin{array}{|c|c|}
\hline
\text{Height, } x & \text{Weight, } y \\
\hline
62 & 136.2 \\
64 & 140.6 \\
66 & 145.6 \\
68 & 151.2 \\
70 & 157.4 \\
72 & 164.2 \\
74 & 171.5 \\
76 & 179.4 \\
\hline
\end{array}
\]

\[\text{FIGURE 1.13}\]

To confirm algebraically the estimate found in part (b), you can substitute 71 for \( x \) in the model.

\[
y = 0.073(71)^2 - 6.99(71) + 289.0 = 160.70
\]

So, the graphical estimate of 161 pounds is fairly good.

CHECKPOINT  Now try Exercise 67.
1.1 Exercises

VOCABULARY CHECK: Fill in the blanks.
1. An ordered pair \((a, b)\) is a ________ of an equation in \(x\) and \(y\) if the equation is true when \(a\) is substituted for \(x\) and \(b\) is substituted for \(y\).
2. The set of all solution points of an equation is the ________ of the equation.
3. The points at which a graph intersects or touches an axis are called the ________ of the graph.
4. A graph is symmetric with respect to the ________ if, whenever \((x, y)\) is on the graph, \((-x, y)\) is also on the graph.
5. The equation \((x - h)^2 + (y - k)^2 = r^2\) is the standard form of the equation of a ________ with center ________ and radius ________.
6. When you construct and use a table to solve a problem, you are using a ________ approach.


In Exercises 1–4, determine whether each point lies on the graph of the equation.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Points</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (y = \sqrt{x + 4})</td>
<td>(a) (0, 2) (b) (5, 3)</td>
<td></td>
</tr>
<tr>
<td>2. (y = x^2 - 3x + 2)</td>
<td>(a) (2, 0) (b) (−2, 8)</td>
<td></td>
</tr>
<tr>
<td>3. (y = 4 -</td>
<td>x - 2</td>
<td>)</td>
</tr>
<tr>
<td>4. (y = \frac{1}{2}x^3 - 2x^2)</td>
<td>(a) (\left(2, -\frac{16}{3}\right)) (b) ((-3, 9))</td>
<td></td>
</tr>
</tbody>
</table>

In Exercises 5–8, complete the table. Use the resulting solution points to sketch the graph of the equation.

5. \(y = -2x + 5\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>( \frac{5}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. \(y = \frac{3}{2}x - 1\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-2)</th>
<th>0</th>
<th>1</th>
<th>4/3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. \(y = x^2 - 3x\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

8. \(y = 5 - x^2\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>−2</th>
<th>−1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

|(x, y)| | | | | |

In Exercises 9–12, find the \(x\)- and \(y\)-intercepts of the graph of the equation.

9. \(y = 16 - 4x^2\)

10. \(y = (x + 3)^2\)

11. \(y = 2x^3 - 4x^2\)

12. \(y^2 = x + 1\)
In Exercises 13–16, assume that the graph has the indicated type of symmetry. Sketch the complete graph of the equation. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.


| y-axis symmetry | x-axis symmetry |

15. 16.  

| origin symmetry | y-axis symmetry |

In Exercises 17–24, use the algebraic tests to check for symmetry with respect to both axes and the origin.

17. \( x^2 - y = 0 \)  
18. \( x - y^2 = 0 \)  
19. \( y = x^3 \)  
20. \( y = x^4 - x^2 + 3 \)  
21. \( y = \frac{x}{x^2 + 1} \)  
22. \( y = \frac{1}{x^2 + 1} \)  
23. \( xy^2 + 10 = 0 \)  
24. \( xy = 4 \)  

In Exercises 25–36, use symmetry to sketch the graph of the equation.

25. \( y = -3x + 1 \)  
26. \( y = 2x - 3 \)  
27. \( y = x^2 - 2x \)  
28. \( y = -x^2 - 2x \)  
29. \( y = x^3 + 3 \)  
30. \( y = x^3 - 1 \)  
31. \( y = \sqrt{x - 3} \)  
32. \( y = \sqrt{1 - x} \)  
33. \( y = |x - 6| \)  
34. \( y = 1 - |x| \)  
35. \( x = y^2 - 1 \)  
36. \( x = y^2 - 5 \)  

In Exercises 37–48, use a graphing utility to graph the equation. Use a standard setting. Approximate any intercepts.

37. \( y = 3 - \frac{2}{3}x \)  
38. \( y = \frac{2}{3}x - 1 \)  
39. \( y = x^2 - 4x + 3 \)  
40. \( y = x^2 + x - 2 \)  
41. \( y = \frac{2x}{x - 1} \)  
42. \( y = \frac{4}{x^2 + 1} \)  
43. \( y = \sqrt[3]{x} \)  
44. \( y = \sqrt[3]{x + 1} \)  

45. \( y = x\sqrt{x} + 6 \)  
46. \( y = (6 - x)\sqrt{x} \)  
47. \( y = |x + 3| \)  
48. \( y = 2 - |x| \)  

In Exercises 49–56, write the standard form of the equation of the circle with the given characteristics.

49. Center: \((0, 0)\); radius: 4  
50. Center: \((0, 0)\); radius: 5  
51. Center: \((2, -1)\); radius: 4  
52. Center: \((-7, -4)\); radius: 7  
53. Center: \((-1, 2)\); solution point: \((0, 0)\)  
54. Center: \((3, -2)\); solution point: \((-1, 1)\)  
55. Endpoints of a diameter: \((0, 0), (6, 8)\)  
56. Endpoints of a diameter: \((-4, -1), (4, 1)\)  

In Exercises 57–62, find the center and radius of the circle, and sketch its graph.

57. \( x^2 + y^2 = 25 \)  
58. \( x^2 + y^2 = 16 \)  
59. \( (x - 1)^2 + (y + 3)^2 = 9 \)  
60. \( x^2 + (y - 1)^2 = 1 \)  
61. \( (x - \frac{1}{2})^2 + (y - \frac{3}{2})^2 = \frac{9}{4} \)  
62. \( (x - 2)^2 + (y + 3)^2 = \frac{16}{9} \)  

63. **Depreciation**  
A manufacturing plant purchases a new molding machine for $225,000. The depreciated value \( y \) (drop in value) after \( t \) years is given by \( y = 225,000 - 20,000t \), \( 0 \leq t \leq 8 \). Sketch the graph of the equation.

64. **Consumerism**  
You purchase a jet ski for $8100. The depreciated value \( y \) after \( t \) years is given by \( y = 8100 - 929t \), \( 0 \leq t \leq 6 \). Sketch the graph of the equation.

65. **Geometry**  
A regulation NFL playing field (including the end zones) of length \( x \) and width \( y \) has a perimeter of \( 336 \frac{2}{3} \) or \( \frac{1040}{3} \) yards.

(a) Draw a rectangle that gives a visual representation of the problem. Use the specified variables to label the sides of the rectangle.

(b) Show that the width of the rectangle is \( y = \frac{520}{3} - x \) and its area is \( A = x\left(\frac{520}{3} - x\right) \).

(c) Use a graphing utility to graph the area equation. Be sure to adjust your window settings.

(d) From the graph in part (c), estimate the dimensions of the rectangle that yield a maximum area.

(e) Use your school’s library, the Internet, or some other reference source to find the actual dimensions and area of a regulation NFL playing field and compare your findings with the results of part (d).
66. **Geometry**  A soccer playing field of length \( x \) and width \( y \) has a perimeter of 360 meters.

(a) Draw a rectangle that gives a visual representation of the problem. Use the specified variables to label the sides of the rectangle.

(b) Show that the width of the rectangle is \( w = 180 - x \) and its area is \( A = x(180 - x) \).

(c) Use a graphing utility to graph the area equation. Be sure to adjust your window settings.

(d) From the graph in part (c), estimate the dimensions of the rectangle that yield a maximum area.

(e) Use your school’s library, the Internet, or some other reference source to find the actual dimensions and area of a regulation Major League Soccer field and compare your findings with the results of part (d).

68. **Electronics**  The resistance \( y \) (in ohms) of 1000 feet of solid copper wire at 68 degrees Fahrenheit can be approximated by the model \( y = \frac{10,770}{x^2} - 0.37 \), \( 5 \leq x \leq 100 \) where \( x \) is the diameter of the wire in mils (0.001 inch).

(Source: American Wire Gage)

(a) Complete the table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Use the table of values in part (a) to sketch a graph of the model. Then use your graph to estimate the resistance when \( x = 85.5 \).

(c) Use the model to confirm algebraically the estimate you found in part (b).

(d) What can you conclude in general about the relationship between the diameter of the copper wire and the resistance?

### Model It

67. **Population Statistics**  The table shows the life expectancies of a child (at birth) in the United States for selected years from 1920 to 2000. (Source: U.S. National Center for Health Statistics)

<table>
<thead>
<tr>
<th>Year</th>
<th>Life expectancy, ( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1920</td>
<td>54.1</td>
</tr>
<tr>
<td>1930</td>
<td>59.7</td>
</tr>
<tr>
<td>1940</td>
<td>62.9</td>
</tr>
<tr>
<td>1950</td>
<td>68.2</td>
</tr>
<tr>
<td>1960</td>
<td>69.7</td>
</tr>
<tr>
<td>1970</td>
<td>70.8</td>
</tr>
<tr>
<td>1980</td>
<td>73.7</td>
</tr>
<tr>
<td>1990</td>
<td>75.4</td>
</tr>
<tr>
<td>2000</td>
<td>77.0</td>
</tr>
</tbody>
</table>

A model for the life expectancy during this period is \( y = -0.0025t^2 + 0.574t + 44.25 \), \( 20 \leq t \leq 100 \) where \( y \) represents the life expectancy and \( t \) is the time in years, with \( t = 20 \) corresponding to 1920.

(a) Sketch a scatter plot of the data.

(b) Graph the model for the data and compare the scatter plot and the graph.

(c) Determine the life expectancy in 1948 both graphically and algebraically.

(d) Use the graph of the model to estimate the life expectancies of a child for the years 2005 and 2010.

(e) Do you think this model can be used to predict the life expectancy of a child 50 years from now? Explain.

### Synthesis

**True or False?** In Exercises 69 and 70, determine whether the statement is true or false. Justify your answer.

69. A graph is symmetric with respect to the \( x \)-axis if, whenever \((x, y)\) is on the graph, \((-x, y)\) is also on the graph.

70. A graph of an equation can have more than one \( y \)-intercept.

### Think About It

71. Suppose you correctly enter an expression for the variable \( y \) on a graphing utility. However, no graph appears on the display when you graph the equation. Give a possible explanation and the steps you could take to remedy the problem. Illustrate your explanation with an example.

72. Find \( a \) and \( b \) if the graph of \( y = ax^2 + bx^3 \) is symmetric with respect to (a) the \( y \)-axis and (b) the origin. (There are many correct answers.)

### Skills Review

73. Identify the terms: \( 9x^5 + 4x^3 - 7 \).

74. Rewrite the expression using exponential notation.

\[ -(7 \times 7 \times 7 \times 7) \]

In Exercises 75–78, simplify the expression.

75. \( \sqrt{18x} - \sqrt{2x} \)

76. \( \frac{55}{\sqrt{20} - 3} \)

77. \( \frac{y}{r^2} \)

78. \( \sqrt{\frac{y}{9}} \)
### Equations and Solutions of Equations

An equation in $x$ is a statement that two algebraic expressions are equal. For example,

$$3x - 5 = 7, \quad x^2 - x - 6 = 0, \quad \text{and} \quad \sqrt{2}x = 4$$

are equations. To solve an equation in $x$ means to find all values of $x$ for which the equation is true. Such values are solutions. For instance, $x = 4$ is a solution of the equation

$$3x - 5 = 7$$

because $3(4) - 5 = 7$ is a true statement.

The solutions of an equation depend on the kinds of numbers being considered. For instance, in the set of rational numbers, $x^2 = 10$ has no solution because there is no rational number whose square is 10. However, in the set of real numbers, the equation has the two solutions $x = \sqrt{10}$ and $x = -\sqrt{10}$.

An equation that is true for every real number in the domain of the variable is called an identity. For example

$$x^2 - 9 = (x + 3)(x - 3) \quad \text{Identity}$$

is an identity because it is a true statement for any real value of $x$. The equation

$$\frac{x}{3x^2} = \frac{1}{3x} \quad \text{Identity}$$

where $x \neq 0$, is an identity because it is true for any nonzero real value of $x$.

An equation that is true for just some (or even none) of the real numbers in the domain of the variable is called a conditional equation. For example, the equation

$$x^2 - 9 = 0 \quad \text{Conditional equation}$$

is conditional because $x = 3$ and $x = -3$ are the only values in the domain that satisfy the equation. The equation $2x - 4 = 2x + 1$ is conditional because there are no real values of $x$ for which the equation is true. Learning to solve conditional equations is the primary focus of this chapter.

### Linear Equations in One Variable

#### Definition of a Linear Equation

A **linear equation in one variable** $x$ is an equation that can be written in the standard form

$$ax + b = 0$$

where $a$ and $b$ are real numbers with $a \neq 0$. 

---

**What you should learn**

- Identify different types of equations.
- Solve linear equations in one variable.
- Solve equations that lead to linear equations.
- Find $x$- and $y$-intercepts of graphs of equations algebraically.
- Use linear equations to model and solve real-life problems.

**Why you should learn it**

Linear equations are used in many real-life applications. For example, in Exercise 98 on page 97, linear equations can be used to model the relationship of the number of married women in the civilian workforce over time.
A linear equation has exactly one solution. To see this, consider the following steps. (Remember that \(a \neq 0\).)

\[
\begin{align*}
ax + b &= 0 & \text{Write original equation.} \\
ax &= -b & \text{Subtract } b \text{ from each side.} \\
x &= -\frac{b}{a} & \text{Divide each side by } a.
\end{align*}
\]

To solve a conditional equation in \(x\) on one side of the equation by a sequence of equivalent (and usually simpler) equations, each having the same solution(s) as the original equation. The operations that yield equivalent equations come from the Substitution Principle and the Properties of Equality studied in Chapter P.

### Generating Equivalent Equations

An equation can be transformed into an equivalent equation by one or more of the following steps.

1. Remove symbols of grouping, combine like terms, or simplify fractions on one or both sides of the equation.
   
   \[
   \begin{align*}
   2x - x &= 4 & \text{Equivalent Equation} \\
   x &= 4
   \end{align*}
   \]

2. Add (or subtract) the same quantity to (from) each side of the equation.
   
   \[
   \begin{align*}
   x + 1 &= 6 & \text{Equivalent Equation} \\
   x &= 5
   \end{align*}
   \]

3. Multiply (or divide) each side of the equation by the same nonzero quantity.
   
   \[
   \begin{align*}
   2x &= 6 & \text{Equivalent Equation} \\
   x &= 3
   \end{align*}
   \]

4. Interchange the two sides of the equation.
   
   \[
   \begin{align*}
   2 &= x & \text{Equivalent Equation} \\
   x &= 2
   \end{align*}
   \]

### Example 1

**Solving a Linear Equation**

a. \(3x - 6 = 0\)  

   \[
   \begin{align*}
   3x &= 6 & \text{Add 6 to each side.} \\
   x &= 2 & \text{Divide each side by 3.}
   \end{align*}
   \]

b. \(5x + 4 = 3x - 8\)  

   \[
   \begin{align*}
   2x + 4 &= -8 & \text{Subtract } 3x \text{ from each side.} \\
   2x &= -12 & \text{Subtract 4 from each side.} \\
   x &= -6 & \text{Divide each side by 2.}
   \end{align*}
   \]

Now try Exercise 27.
After solving an equation, you should check each solution in the original equation. For instance, you can check the solution to Example 1(a) as follows.

\[
\begin{align*}
3x - 6 &= 0 & \text{Write original equation.} \\
3(2) - 6 &= 0 & \text{Substitute 2 for } x. \\
0 &= 0 & \text{Solution checks. ✓}
\end{align*}
\]

Try checking the solution to Example 1(b).

Some equations have no solutions because all the \(x\)-terms sum to zero and a contradictory (false) statement such as \(0 = 5\) or \(12 = 7\) is obtained. For instance, the equation

\[
x = x + 1
\]

has no solution. Watch for this type of equation in the exercises.

### Example 2 Solving a Linear Equation

Solve

\[
6(x - 1) + 4 = 3(7x + 1).
\]

#### Solution

\[
\begin{align*}
6(x - 1) + 4 &= 3(7x + 1) & \text{Write original equation.} \\
6x - 6 + 4 &= 21x + 3 & \text{Distributive Property} \\
6x - 2 &= 21x + 3 & \text{Simplify.} \\
-15x - 2 &= 3 & \text{Subtract } 21x \text{ from each side.} \\
-15x &= 5 & \text{Add 2 to each side.} \\
x &= -\frac{1}{3} & \text{Divide each side by } -15.
\end{align*}
\]

#### Check

Check this solution by substituting \(-\frac{1}{3}\) for \(x\) in the original equation.

\[
\begin{align*}
6(x - 1) + 4 &= 3(7x + 1) & \text{Write original equation.} \\
6\left(-\frac{1}{3} - 1\right) + 4 &= 3\left(7\left(-\frac{1}{3}\right) + 1\right) & \text{Substitute } -\frac{1}{3} \text{ for } x. \\
6\left(-\frac{4}{3}\right) + 4 &= 3\left(-\frac{2}{3} + 1\right) & \text{Simplify.} \\
6\left(-\frac{4}{3}\right) + 4 &= 3\left(-\frac{4}{3}\right) & \text{Simplify.} \\
-\frac{24}{3} + 4 &= -\frac{12}{3} & \text{Multiply.} \\
-8 + 4 &= -4 & \text{Simplify.} \\
-4 &= -4 & \text{Solution checks. ✓}
\end{align*}
\]

So, the solution is \(x = -\frac{1}{3}\). Note that if you subtracted 6\(x\) from each side of the equation and then subtracted 3 from each side of the equation, you would still obtain the solution \(x = -\frac{1}{3}\).

🤝 **CHECKPOINT** Now try Exercise 29.
Equations That Lead to Linear Equations

To solve an equation involving fractional expressions, find the least common denominator (LCD) of all terms and multiply every term by the LCD. This process will clear the original equation of fractions and produce a simpler equation to work with.

Example 3  An Equation Involving Fractional Expressions

Solve \( \frac{x}{3} + \frac{3x}{4} = 2 \).

Solution

Write original equation.

\[
\frac{x}{3} + \frac{3x}{4} = 2
\]

Multiply each term by the LCD of 12.

\[
(12) \frac{x}{3} + (12) \frac{3x}{4} = (12)2
\]

Divide out and multiply.

\[
4x + 9x = 24
\]

Combine like terms.

\[
13x = 24
\]

Divide each side by 13.

\[
x = \frac{24}{13}
\]

The solution is \( x = \frac{24}{13} \). Check this in the original equation.

Now try Exercise 33.

When multiplying or dividing an equation by a variable quantity, it is possible to introduce an extraneous solution. An extraneous solution is one that does not satisfy the original equation. Therefore, it is essential that you check your solutions.

Example 4  An Equation with an Extraneous Solution

Solve \( \frac{1}{x - 2} = \frac{3}{x + 2} - \frac{6x}{x^2 - 4} \).

Solution

The LCD is \( x^2 - 4 \), or \( (x + 2)(x - 2) \). Multiply each term by this LCD.

\[
\frac{1}{x - 2} (x + 2)(x - 2) = \frac{3}{x + 2} (x + 2)(x - 2) - \frac{6x}{x^2 - 4} (x + 2)(x - 2)
\]

\[
x + 2 = 3(x - 2) - 6x, \quad x \neq \pm 2
\]

\[
x + 2 = 3x - 6 - 6x
\]

\[
x + 2 = -3x - 6
\]

\[
4x = -8 \quad \rightarrow \quad x = -2 \quad \text{Extraneous solution}
\]

In the original equation, \( x = -2 \) yields a denominator of zero. So, \( x = -2 \) is an extraneous solution, and the original equation has no solution.

Now try Exercise 53.
Finding Intercepts Algebraically

In Section 1.1, you learned to find \( x \)- and \( y \)-intercepts using a graphical approach. Because all the points on the \( x \)-axis have a \( y \)-coordinate equal to zero, and all the points on the \( y \)-axis have an \( x \)-coordinate equal to zero, you can use an algebraic approach to find \( x \)- and \( y \)-intercepts, as follows.

Here is an example.

\[
y = 4x + 1 \implies 0 = 4x + 1 \implies -1 = 4x \implies -\frac{1}{4} = x
\]

So, the \( x \)-intercept of \( y = 4x + 1 \) is \((-\frac{1}{4}, 0)\) and the \( y \)-intercept is \((0, 1)\).

Application

Example 5  Female Participants in Athletic Programs

The number \( y \) (in millions) of female participants in high school athletic programs (in millions) in the United States from 1989 to 2002 can be approximated by the linear model

\[
y = 0.085t + 1.83, \quad -1 \leq t \leq 12
\]

where \( t = 0 \) represents 1990. (a) Find the \( y \)-intercept of the graph of the linear model shown in Figure 1.14 algebraically. (b) Assuming that this linear pattern continues, find the year in which there will be 3.36 million female participants. (Source: National Federation of State High School Associations)

Solution

a. To find the \( y \)-intercept, let \( t = 0 \) and solve for \( y \) as follows.

\[
y = 0.085(0) + 1.83 \\
= 1.83
\]

So, the \( y \)-intercept is \((0, 1.83)\).

b. Let \( y = 3.36 \) and solve the equation \( 3.36 = 0.085t + 1.83 \) for \( t \).

\[
3.36 = 0.085t + 1.83 \\
1.53 = 0.085t \\
18 = t
\]

Because \( t = 0 \) represents 1990, \( t = 18 \) must represent 2008. So, from this model, there will be 3.36 million female participants in 2008.

Now try Exercise 97.
1.2 Exercises

VOCABULARY CHECK: Fill in the blanks.
1. An ________ is a statement that equates two algebraic expressions.
2. To find all values that satisfy an equation is to ________ the equation.
3. There are two types of equations, ________ and ________ equations.
4. A linear equation in one variable is an equation that can be written in the standard form ________.
5. When solving an equation, it is possible to introduce an ________ solution, which is a value that does not satisfy the original equation.


In Exercises 1–10, determine whether each value of \( x \) is a solution of the equation.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( 5x - 3 = 3x + 5 )</td>
<td>(a) ( x = 0 ) (b) ( x = -5 ) (c) ( x = 4 ) (d) ( x = 10 )</td>
</tr>
<tr>
<td>2. ( 7 - 3x = 5x - 17 )</td>
<td>(a) ( x = -3 ) (b) ( x = 8 ) (c) ( x = 4 ) (d) ( x = 3 )</td>
</tr>
<tr>
<td>3. ( 3x^2 + 2x - 5 = 2x^2 - 2 )</td>
<td>(a) ( x = -3 ) (b) ( x = 1 ) (c) ( x = 4 ) (d) ( x = -5 )</td>
</tr>
<tr>
<td>4. ( 5x^3 + 2x - 3 = 4x^3 + 2x - 11 )</td>
<td>(a) ( x = 2 ) (b) ( x = -2 ) (c) ( x = 0 ) (d) ( x = 10 )</td>
</tr>
<tr>
<td>5. ( \frac{5}{2x} = \frac{4}{x} = 3 )</td>
<td>(a) ( x = -1/2 ) (b) ( x = 4 ) (c) ( x = 0 ) (d) ( x = 1/2 )</td>
</tr>
<tr>
<td>6. ( 3 + \frac{1}{x + 2} = 4 )</td>
<td>(a) ( x = -1 ) (b) ( x = -2 ) (c) ( x = 0 ) (d) ( x = 5 )</td>
</tr>
<tr>
<td>7. ( \sqrt{3x - 2} = 4 )</td>
<td>(a) ( x = 3 ) (b) ( x = 2 ) (c) ( x = 9 ) (d) ( x = -6 )</td>
</tr>
<tr>
<td>8. ( \frac{3}{x - 8} = 3 )</td>
<td>(a) ( x = 2 ) (b) ( x = -5 ) (c) ( x = 35 ) (d) ( x = 8 )</td>
</tr>
<tr>
<td>9. ( 6x^2 - 11x - 35 = 0 )</td>
<td>(a) ( x = -5/2 ) (b) ( x = -7/2 ) (c) ( x = 7/2 ) (d) ( x = 5/2 )</td>
</tr>
<tr>
<td>10. ( 10x^2 + 21x - 10 = 0 )</td>
<td>(a) ( x = 5/2 ) (b) ( x = -3/2 ) (c) ( x = -1/2 ) (d) ( x = -2 )</td>
</tr>
</tbody>
</table>

In Exercises 11–20, determine whether the equation is an identity or a conditional equation.

11. \( 2(x - 1) = 2x - 2 \)  12. \( 3(x + 2) = 5x + 4 \)
13. \( -6(x - 3) + 5 = -2x + 10 \)
14. \( 3(x + 2) - 5 = 3x + 1 \)
15. \( 4(x + 1) - 2x = 2(x + 2) \)
16. \( -7(x - 3) + 4x = 3(7 - x) \)
17. \( x^2 - 8x + 5 = (x - 4)^2 - 11 \)
18. \( x^2 + 2(3x - 2) = x^2 + 6x - 4 \)
19. \( 3 + \frac{1}{x + 1} = \frac{4x}{x + 1} \)
20. \( \frac{5}{x} + \frac{3}{x} = 24 \)

In Exercises 21 and 22, justify each step of the solution.

21. \( 4x + 32 = 83 \)
   \( 4x + 32 - 32 = 83 - 32 \)
   \( 4x = 51 \)
   \( x = \frac{51}{4} \)
22. \( 3(x - 4) + 10 = 7 \)
   \( 3x - 12 + 10 = 7 \)
   \( 3x - 2 = 7 \)
   \( 3x = 9 \)
   \( x = 3 \)

In Exercises 23–38, solve the equation and check your solution.

23. \( x + 11 = 15 \)
24. \( 7 - x = 19 \)
25. \( 7 - 2x = 25 \)
26. \( 7x + 2 = 23 \)
27. \( 8x - 5 = 3x + 20 \)
28. \( 7x + 3 = 3x - 17 \)
29. \( 2(x + 5) - 7 = 3(x - 2) \)
30. \( 3(x + 3) = 5(1 - x) - 1 \)
31. \( x - 3(2x + 3) = 8 - 5x \)
32. $9x - 10 = 5x + 2(2x - 5)$
33. $\frac{5x}{3} + \frac{1}{4} = x - \frac{1}{2}$
34. $\frac{x}{5} - \frac{x}{2} = 3 + \frac{3x}{10}$
35. $\frac{3}{2}(z + 5) - \frac{1}{3}(z + 24) = 0$
36. $\frac{3x}{2} + \frac{1}{4}(x - 2) = 10$
37. $0.25x + 0.75(10 - x) = 3$
38. $0.60x + 0.40(100 - x) = 50$

In Exercises 39–42, solve the equation using two different methods. Then explain which method is easier.
39. $3(x - 1) = 4$
40. $4(x + 3) = 15$
41. $\frac{1}{3}(x + 2) = 5$
42. $\frac{3}{5}(z - 4) = 6$

In Exercises 43–64, solve the equation and check your solution. (If not possible, explain why.)
43. $x + 8 = 2(x - 2) - x$
44. $8(x + 2) - 3(2x + 1) = 2(x + 5)$
45. $\frac{100 - 4x}{3} = \frac{5x + 6}{4} + 6$
46. $\frac{17 + y}{y} + \frac{32 + y}{y} = 100$
47. $\frac{5x - 4}{5x + 4} = \frac{2}{3}$
48. $\frac{10x + 3}{5x + 6} = \frac{1}{2}$
49. $10 - \frac{13}{x} = 4 + \frac{5}{x}$
50. $\frac{15}{x} - 4 = \frac{6}{x} + 3$
51. $3 = 2 + \frac{2}{x + 2}$
52. $\frac{1}{x} + \frac{2}{x - 5} = 0$
53. $\frac{x}{x + 4} + \frac{4}{x + 4} + 2 = 0$
54. $\frac{7}{2x + 1} - \frac{8x}{2x - 1} = -4$
55. $\frac{2}{(x - 4)(x - 2)} = \frac{1}{x - 4} + \frac{2}{x - 2}$
56. $\frac{4}{x - 1} + \frac{6}{3x + 1} = \frac{15}{3x + 1}$
57. $\frac{1}{x - 3} + \frac{1}{x + 3} = \frac{10}{x^2 - 9}$
58. $\frac{1}{x - 2} + \frac{3}{x + 3} = \frac{4}{x^2 + x - 6}$
59. $\frac{3}{x^2 - 3x} + \frac{4}{x} = \frac{1}{x - 3}$
60. $\frac{6}{x} - \frac{2}{x + 3} = \frac{3(x + 5)}{x^2 + 3x}$
61. $(x + 2)^2 + 5 = (x + 3)^2$
62. $(x + 1)^2 + 2(x - 2) = (x + 1)(x - 2)$
63. $(x + 2)^2 - x^2 = 4(x + 1)$
64. $(2x + 1)^2 = 4(x^2 + x + 1)$

Graphical Analysis In Exercises 65–70, use a graphing utility to graph the equation and approximate any x-intercepts. Set y = 0 and solve the resulting equation. Compare the results with the graph's x-intercepts.
65. $y = 2(x - 1) - 4$
66. $y = \frac{3}{5}x + 2$
67. $y = 20 - (3x - 10)$
68. $y = 10 + 2(x - 2)$
69. $y = -38 + 5(9 - x)$
70. $y = 6x - 6(\frac{16}{\pi} + x)$

In Exercises 71–80, find the x- and y-intercepts of the graph of the equation algebraically.
71. $y = 12 - 5x$
72. $y = 16 - 3x$
73. $y = -3(2x + 1)$
74. $y = 5 - (6 - x)$
75. $2x + 3y = 10$
76. $4x - 5y = 12$
77. $\frac{2x}{3} + 8 - 3y = 0$
78. $\frac{8x}{3} + 5 - 2y = 0$
79. $4y - 0.75x + 1.2 = 0$
80. $3y + 2.5x - 3.4 = 0$

In Exercises 81–88, solve for $x$.
81. $4(x + 1) - ax = x + 5$
82. $4 - 2(x - 2b) = ax + 3$
83. $6x + ax = 2x + 5$
84. $5 + ax = 12 - bx$
85. $19x + \frac{1}{3}ax = x + 9$
86. $-5(3x - 6b) + 12 = 8 + 3ax$
87. $-2ax + 6(x + 3) = -4x + 1$
88. $\frac{4}{3}x - ax = 2(\frac{2}{3}x - 1) + 10$

In Exercises 89–92, solve the equation for $x$. (Round your solution to three decimal places.)
89. $0.275x + 0.725(500 - x) = 300$
90. $2.763 - 4.5(2.1x - 5.1432) = 6.32x + 5$
91. $\frac{2}{7.398} - 4.405 = \frac{1}{x}$
92. $\frac{3}{6.350} - \frac{6}{x} = 18$

93. Geometry The surface area $S$ of the circular cylinder shown in the figure is

$$S = 2\pi(25) + 2\pi(5h).$$

Find the height $h$ of the cylinder if the surface area is 471 square feet. Use 3.14 for $\pi$. 

![Diagram of a cylinder](image-url)
94. **Geometry**  The surface area \( S \) of the rectangular solid in the figure is \( S = 2(24) + 2(4x) + 2(6x) \). Find the length \( x \) of the box if the surface area is 248 square centimeters.

![Diagram of a rectangular solid]

95. **Anthropology**  The relationship between the length of an adult’s femur (thigh bone) and the height of the adult can be approximated by the linear equations

\[
\begin{align*}
\text{Female:} & \quad y = 0.432x - 10.44 \\
\text{Male:} & \quad y = 0.449x - 12.15
\end{align*}
\]

where \( y \) is the length of the femur in inches and \( x \) is the height of the adult in inches (see figure).

(a) An anthropologist discovers a femur belonging to an adult human female. The bone is 16 inches long. Estimate the height of the female.

(b) From the foot bones of an adult human male, an anthropologist estimates that the person’s height was 69 inches. A few feet away from the site where the foot bones were discovered, the anthropologist discovers a male adult femur that is 19 inches long. Is it likely that both the foot bones and the thigh bone came from the same person?

(c) Complete the table to determine if there is a height of an adult for which an anthropologist would not be able to determine whether the femur belonged to a male or a female.

<table>
<thead>
<tr>
<th>( x )</th>
<th>Female femur length, ( y )</th>
<th>Male femur length, ( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>70</td>
<td></td>
<td></td>
</tr>
<tr>
<td>80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>90</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>110</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(d) Solve part (c) algebraically by setting the two equations equal to each other and solving for \( x \). Compare your solutions. Do you believe an anthropologist would ever have the problem of not being able to determine whether a femur belonged to a male or a female? Why or why not?

96. **Tax Credits**  Use the following information about a possible tax credit for a family consisting of two adults and two children (see figure).

**Table:**

<table>
<thead>
<tr>
<th>Earned income (in thousands of dollars)</th>
<th>Total income (T)</th>
<th>Subsidy (S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td>10</td>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td>12</td>
<td>14</td>
<td>18</td>
</tr>
<tr>
<td>14</td>
<td>16</td>
<td>20</td>
</tr>
</tbody>
</table>

Earned income: \( E \)

Subsidy (a grant of money): \( S = 10,000 - \frac{1}{2}E \), \( 0 \leq E \leq 20,000 \)

Total income: \( T = E + S \)

(a) Write the total income \( T \) in terms of \( E \).

(b) Find the earned income \( E \) if the subsidy is \$6600.

(c) Find the earned income \( E \) if the total income is \$13,800.

(d) Find the subsidy \( S \) if the total income is \$12,500.
97. **Fuel Consumption** The annual consumption $y$ of gasoline (in billions of gallons) by vans, pickup trucks, and SUVs in the United States from 1989 to 2002 can be approximated by the model

$$y = 1.64t + 36.8, \quad -1 \leq t \leq 12$$

where $t$ represents the year, with $t = 0$ corresponding to 1990.

(a) Sketch a graph of the model. Graphically estimate the y-intercept of the graph.

(b) Find the y-intercept of the graph algebraically.

(c) Assuming this linear pattern continues, find the year in which the annual consumption of gasoline will be 65 billion gallons. Does your answer seem reasonable? Explain.

98. **Labor Statistics** The number of married women $y$ (in millions) in the civilian work force in the United States from 1990 to 2002 (see figure) can be approximated by the model

$$y = 0.39t + 31.0, \quad 0 \leq t \leq 12$$

where $t$ represents the year, with $t = 0$ corresponding to 1990. (Source: U.S. Bureau of Labor Statistics)

(a) According to this model, during which year did the number reach 33 million?

(b) Explain how you can solve part (a) graphically and algebraically.

99. **Operating Cost** A delivery company has a fleet of vans. The annual operating cost $C$ per van is

$$C = 0.32m + 2500$$

where $m$ is the number of miles traveled by a van in a year. What number of miles will yield an annual operating cost of $10,000? 

100. **Flood Control** A river has risen 8 feet above its flood stage. The water begins to recede at a rate of 3 inches per hour. Write a mathematical model that shows the number of feet above flood stage after $t$ hours. If the water continually recedes at this rate, when will the river be 1 foot above its flood stage?

**Synthesis**

**True or False?** In Exercises 101 and 102, determine whether the statement is true or false. Justify your answer.

101. The equation $x(3 - x) = 10$ is a linear equation.

102. The equation $x^2 + 9x - 5 = 4 - x^3$ has no real solution.

103. **Think About It** What is meant by equivalent equations? Give an example of two equivalent equations.

104. **Writing** Describe the steps used to transform an equation into an equivalent equation.

105. **Exploration**

(a) Complete the table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>3.2</th>
<th>5.8</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3.2x - 5.8$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Use the table in part (a) to determine the interval in which the solution to the equation $3.2x - 5.8 = 0$ is located. Explain your reasoning.

(c) Complete the table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1.5</th>
<th>1.6</th>
<th>1.7</th>
<th>1.8</th>
<th>1.9</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3.2x - 5.8$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(d) Use the table in part (c) to determine the interval in which the solution to the equation $3.2x - 5.8 = 0$ is located. Explain how this process can be used to approximate the solution to any desired degree of accuracy.

106. **Exploration** Use the procedure in Exercise 105 to approximate the solution to the equation

$$0.3(x - 1.5) - 2 = 0$$

accurate to two decimal places.

**Skills Review**

In Exercises 107 and 108, simplify the expression.

107. $\frac{x^2 + 5x - 36}{2x^2 + 17x - 9}$

108. $\frac{x^2 - 49}{x^3 + x^2 + 3x - 21}$

In Exercises 109–112, sketch the graph of the equation.

109. $y = 3x - 5$

110. $y = -\frac{1}{2}x - \frac{9}{2}$

111. $y = -x^2 - 5x$

112. $y = \sqrt{5 - x}$
1.3 Modeling with Linear Equations

What you should learn
- Use a verbal model in a problem-solving plan.
- Write and use mathematical models to solve real-life problems.
- Solve mixture problems.
- Use common formulas to solve real-life problems.

Why you should learn it
You can use linear equations to determine the percents of income and of expenses of the federal government that come from various sources. See Exercise 42 on page 106.

Introduction to Problem Solving

In this section, you will learn how algebra can be used to solve problems that occur in real-life situations. The process of translating phrases or sentences into algebraic expressions or equations is called **mathematical modeling**. A good approach to mathematical modeling is to use two stages. Begin by using the verbal description of the problem to form a **verbal model**. Then, after assigning labels to the quantities in the verbal model, form a **mathematical model** or **algebraic equation**.

When you are trying to construct a verbal model, it is helpful to look for a **hidden equality**—a statement that two algebraic expressions are equal.

**Example 1** Using a Verbal Model

You have accepted a job for which your annual salary will be $32,300. This salary includes a year-end bonus of $500. You will be paid twice a month. What will your gross pay (pay before taxes) be for each paycheck?

**Solution**

Because there are 12 months in a year and you will be paid twice a month, it follows that you will receive 24 paychecks during the year. You can construct an algebraic equation for this problem as follows. Begin with a verbal model, then assign labels, and finally form an algebraic equation.

<table>
<thead>
<tr>
<th>Verbal Description</th>
<th>Verbal Model</th>
<th>Algebraic Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Income for year = 24 paychecks + Bonus</td>
<td>Income for year = 32,300 (dollars) Amount of each paycheck = x (dollars) Bonus = 500 (dollars)</td>
<td>32,300 = 24x + 500</td>
</tr>
</tbody>
</table>

The algebraic equation for this problem is a **linear equation** in the variable $x$, which you can solve as follows.

\[
32,300 = 24x + 500 \
32,300 - 500 = 24x + 500 - 500 \
31,800 = 24x \
\frac{31,800}{24} = \frac{24x}{24} \
1325 = x
\]

So, your gross pay for each paycheck will be $1325.

**CHECKPOINT** Now try Exercise 29.
A fundamental step in writing a mathematical model to represent a real-life problem is translating key words and phrases into algebraic expressions and equations. The following list gives several examples.

### Translating Key Words and Phrases

<table>
<thead>
<tr>
<th>Key Words and Phrases</th>
<th>Verbal Description</th>
<th>Algebraic Expression or Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Equality:</strong></td>
<td>Equals, equal to, is, are, was, will be, represents</td>
<td>• The sale price $S$ is $10 less than the list price $L$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$S = L - 10$</td>
</tr>
<tr>
<td><strong>Addition:</strong></td>
<td>Sum, plus, greater than, increased by, more than, exceeds, total of</td>
<td>• The sum of 5 and $x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$5 + x$ or $x + 5$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Seven more than $y$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$7 + y$ or $y + 7$</td>
</tr>
<tr>
<td><strong>Subtraction:</strong></td>
<td>Difference, minus, less than, decreased by, subtracted from, reduced by, the remainder</td>
<td>• The difference of 4 and $b$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$4 - b$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Three less than $z$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$z - 3$</td>
</tr>
<tr>
<td><strong>Multiplication:</strong></td>
<td>Product, multiplied by, twice, times, percent of</td>
<td>• Two times $x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Three percent of $t$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.03t$</td>
</tr>
<tr>
<td><strong>Division:</strong></td>
<td>Quotient, divided by, ratio, per</td>
<td>• The ratio of $x$ to 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{x}{8}$</td>
</tr>
</tbody>
</table>

### Using Mathematical Models

#### Example 2 Finding the Percent of a Raise

You have accepted a job that pays $8 an hour. You are told that after a two-month probationary period, your hourly wage will be increased to $9 an hour. What percent raise will you receive after the two-month period?

**Solution**

**Verbal Model:**

\[
\text{Raise} = \text{Percent} \cdot \text{Old wage}
\]

**Labels:**

- Old wage = 8 (dollars per hour)
- New wage = 9 (dollars per hour)
- Raise = 9 − 8 = 1 (dollars per hour)
- Percent = $r$ (percent in decimal form)

**Equation:**

\[
1 = r \cdot 8
\]

\[
\frac{1}{8} = r \quad \text{Divide each side by 8.}
\]

\[
0.125 = r \quad \text{Rewrite fraction as a decimal.}
\]

You will receive a raise of 0.125 or 12.5%.

**Checkpoint** Now try Exercise 41.
Example 3  Finding the Percent of Monthly Expenses

Your family has an annual income of $57,000 and the following monthly expenses: mortgage ($1100), car payment ($375), food ($295), utilities ($240), and credit cards ($220). The total value of the monthly expenses represents what percent of your family’s annual income?

Solution
The total amount of your family’s monthly expenses is $2230. The total monthly expenses for 1 year are $26,760.

Verbal Model:
Labels:
Income (dollars)  Monthly expenses = Percent • Income (dollars)
Percent (in decimal form)

Equation:
\[
26,760 = r \cdot 57,000
\]
\[
\frac{26,760}{57,000} = r
\]
Divide each side by 57,000.
\[
0.469 \approx r
\]
Use a calculator.

Your family’s monthly expenses are approximately 0.469 or 46.9% of your family’s annual income.

Example 4  Finding the Dimensions of a Room

A rectangular kitchen is twice as long as it is wide, and its perimeter is 84 feet. Find the dimensions of the kitchen.

Solution
For this problem, it helps to sketch a diagram, as shown in Figure 1.15.

Verbal Model:
Labels:
Perimeter = 84 (feet)
Width = w (feet)
Length = l = 2w (feet)

Equation:
\[
2(2w) + 2w = 84
\]
\[
6w = 84
\]
\[
w = 14
\]
Group like terms.
Divide each side by 6.

Because the length is twice the width, you have
\[
l = 2w
\]
Length is twice width.
\[
= 2(14) = 28
\]
Substitute and simplify.

So, the dimensions of the room are 14 feet by 28 feet.

CHECKPOINT  Now try Exercise 47.
Example 5  A Distance Problem

A plane is flying nonstop from Atlanta to Portland, a distance of about 2700 miles, as shown in Figure 1.16. After 1.5 hours in the air, the plane flies over Kansas City (a distance of 820 miles from Atlanta). Estimate the time it will take the plane to fly from Atlanta to Portland.

Solution

Verbal Model: \[ \text{Distance} = \text{Rate} \cdot \text{Time} \]

Labels: \[
\begin{align*}
\text{Distance} &= 2700 \text{ miles} \\
\text{Time} &= t \text{ hours} \\
\text{Rate} &= \frac{\text{distance to Kansas City}}{\text{time to Kansas City}} = \frac{820}{1.5} \text{ miles per hour}
\end{align*}
\]

Equation: \[ 2700 = \frac{820}{1.5}t \]

\[ 4050 = 820t \]

\[ \frac{4050}{820} = t \]

\[ 4.94 = t \]

The trip will take about 4.94 hours, or about 4 hours and 56 minutes.

CHECKPOINT  Now try Exercise 53.

Example 6  An Application Involving Similar Triangles

To determine the height of the Aon Center Building (in Chicago), you measure the shadow cast by the building and find it to be 142 feet long, as shown in Figure 1.17. Then you measure the shadow cast by a four-foot post and find it to be 6 inches long. Estimate the building’s height.

Solution

To solve this problem, you use a result from geometry that states that the ratios of corresponding sides of similar triangles are equal.

Verbal Model: \[ \frac{\text{Height of building}}{\text{Length of building’s shadow}} = \frac{\text{Height of post}}{\text{Length of post’s shadow}} \]

Labels: \[
\begin{align*}
\text{Height of building} &= x \text{ feet} \\
\text{Length of building’s shadow} &= 142 \text{ feet} \\
\text{Height of post} &= 4 \text{ feet} = 48 \text{ inches} \\
\text{Length of post’s shadow} &= 6 \text{ inches}
\end{align*}
\]

Equation: \[ \frac{x}{142} = \frac{48}{6} \]

\[ x = 1136 \]

So, the Aon Center Building is about 1136 feet high.

CHECKPOINT  Now try Exercise 65.
Example 7  A Simple Interest Problem

You invested a total of $10,000 at \(\frac{4}{2}\%\) and \(\frac{5}{2}\%\) simple interest. During 1 year, the two accounts earned $508.75. How much did you invest in each account?

Solution

Verbal Model: 

- Interest from \(\frac{4}{2}\%\) = $x \times \frac{4}{2} \times \frac{1}{100} = \frac{1}{2}x\) (dollars)
- Interest from \(\frac{5}{2}\%\) = \(10,000 - x\) \times \frac{5}{2} \times \frac{1}{100} = \frac{5}{2}(10,000 - x)\) (dollars)

Equation: 

\[
0.045x + 0.055(10,000 - x) = 508.75
\]

\[-0.01x = -41.25 \quad \Rightarrow \quad x = 4125\]

So, $4125 was invested at \(\frac{4}{2}\%\) and \(10,000 - x\) or $5875 was invested at \(\frac{5}{2}\%\).

Example 8  An Inventory Problem

A store has $30,000 of inventory in single-disc DVD players and multi-disc DVD players. The profit on a single-disc player is 22\% and the profit on a multi-disc player is 40\%. The profit for the entire stock is 35\%. How much was invested in each type of DVD player?

Solution

Verbal Model: 

- Profit from single-disc players = \(x\) \times 22\% = \(0.22x\) (dollars)
- Profit from multi-disc players = \(30,000 - x\) \times 40\% = \(0.40(30,000 - x)\) (dollars)

Equation: 

\[
0.22x + 0.40(30,000 - x) = 10,500
\]

\[-0.18x = -1500 \quad \Rightarrow \quad x = 8333.33\]

So, $8333.33 was invested in single-disc DVD players and \(30,000 - x\) or $21,666.67 was invested in multi-disc DVD players.

CHECKPOINT Now try Exercise 69.

CHECKPOINT Now try Exercise 63.
Common Formulas

A literal equation is an equation that contains more than one variable. A formula is an example of a literal equation. Many common types of geometric, scientific, and investment problems use ready-made equations called formulas. Knowing these formulas will help you translate and solve a wide variety of real-life applications.

### Common Formulas for Area A, Perimeter P, Circumference C, and Volume V

**Square**

- Area: $A = s^2$
- Perimeter: $P = 4s$

**Rectangle**

- Area: $A = lw$
- Perimeter: $P = 2l + 2w$

**Circle**

- Area: $A = \pi r^2$
- Circumference: $C = 2\pi r$

**Triangle**

- Area: $A = \frac{1}{2}bh$
- Perimeter: $P = a + b + c$

### Miscellaneous Common Formulas

**Temperature:**

- Fahrenheit to Celsius: $F = \frac{9}{5}C + 32$
- Celsius to Fahrenheit: $C = \frac{5}{9}(F - 32)$

**Simple Interest:**

- $I = Prt$
- $I$ = interest, $P$ = principal (original deposit), $r$ = annual interest rate, $t$ = time in years

**Compound Interest:**

- $A = P\left(1 + \frac{r}{n}\right)^{nt}$
- $n$ = compoundings (number of times interest is calculated) per year, $t$ = time in years, $A$ = balance, $P$ = principal (original deposit), $r$ = annual interest rate

**Distance:**

- $d = rt$
- $d$ = distance traveled, $r$ = rate, $t$ = time
When working with applied problems, you often need to rewrite a literal equation in terms of another variable. You can use the methods for solving linear equations to solve some literal equations for a specified variable. For instance, the formula for the perimeter of a rectangle, \( P = 2l + 2w \), can be rewritten or solved for \( w \) as \( w = \frac{1}{2}(P - 2l) \).

### Example 9  Using a Formula

A cylindrical can has a volume of 200 cubic centimeters (cm\(^3\)) and a radius of 4 centimeters (cm), as shown in Figure 1.18. Find the height of the can.

**Solution**

The formula for the volume of a cylinder is \( V = \pi r^2 h \). To find the height of the can, solve for \( h \).

\[
h = \frac{V}{\pi r^2}
\]

Then, using \( V = 200 \) and \( r = 4 \), find the height.

\[
h = \frac{200}{\pi (4)^2} = \frac{200}{16\pi} = 3.98 \quad \text{(Use a calculator)}
\]

You can use unit analysis to check that your answer is reasonable.

\[
\frac{200 \text{ cm}^3}{16\pi \text{ cm}^2} \approx 3.98 \text{ cm}
\]

CHECKPOINT: Now try Exercise 91.

---

**Writing about Mathematics**

Translating Algebraic Formulas Most people use algebraic formulas every day—sometimes without realizing it because they use a verbal form or think of an often-repeated calculation in steps. Translate each of the following verbal descriptions into an algebraic formula, and demonstrate the use of each formula.

**a. Designing Billboards** “The letters on a sign or billboard are designed to be readable at a certain distance. Take half the letter height in inches and multiply by 100 to find the readable distance in feet.”—Thos. Hodgson, Hodgson Signs (Source: Rules of Thumb by Tom Parker)

**b. Percent of Calories from Fat** “To calculate percent of calories from fat, multiply grams of total fat per serving by 9, then divide by the number of calories per serving.” (Source: Good Housekeeping)

**c. Building Stairs** “A set of steps will be comfortable to use if two times the height of one riser plus the width of one tread is equal to 26 inches.”—Alice Lukens Bachelder, gardener (Source: Rules of Thumb by Tom Parker)
1.3 Exercises

VOCABULARY CHECK:
In Exercises 1 and 2, fill in the blanks.
1. The process of translating phrases or sentences into algebraic expressions or equations is called ________ ________.
2. A good approach to mathematical modeling is a two-stage approach, using a verbal description to form a(n) ________ ________, and then after assigning labels to the quantities, form a(n) ________ ________.

In Exercises 3–8, write the formula for the given quantity.
3. Area of a circle: ________
4. Perimeter of a rectangle: ________
5. Volume of a cube: ________
6. Volume of a circular cylinder: ________
7. Balance if $P$ dollars is invested at $r\%$ compounded monthly for $t$ years: ________
8. Simple interest if $P$ dollars is invested at $r\%$ for $t$ years: ________


In Exercises 1–10, write a verbal description of the algebraic expression without using the variable.
1. $x + 4$
2. $t - 10$
3. $\frac{u}{5}$
4. $\frac{2}{3^t}$
5. $\frac{y - 4}{5}$
6. $\frac{z + 10}{7}$
7. $-3(b + 2)$
8. $\frac{-5(x - 1)}{8}$
9. $12x(x - 5)$
10. $\frac{(q + 4)(3 - q)}{2q}$

In Exercises 11–22, write an algebraic expression for the verbal description.
11. The sum of two consecutive natural numbers
12. The product of two consecutive natural numbers
13. The product of two consecutive odd integers, the first of which is $2n - 1$
14. The sum of the squares of two consecutive even integers, the first of which is $2n$
15. The distance traveled in $t$ hours by a car traveling at 50 miles per hour
16. The travel time for a plane traveling at a rate of $r$ kilometers per hour for 200 kilometers
17. The amount of acid in $x$ liters of a 20% acid solution
18. The sale price of an item that is discounted 20% of its list price $L$
19. The perimeter of a rectangle with a width $x$ and a length that is twice the width
20. The area of a triangle with base 20 inches and height $h$ inches
21. The total cost of producing $x$ units for which the fixed costs are $1200 and the cost per unit is $25
22. The total revenue obtained by selling $x$ units at $3.59 per unit

In Exercises 23–26, translate the statement into an algebraic expression or equation.
23. Thirty percent of the list price $L$
24. The amount of water in $q$ quarts of a liquid that is 35% water
25. The percent of 500 that is represented by the number $N$
26. The percent change in sales from one month to the next if the monthly sales are $S_1$ and $S_2$, respectively

In Exercises 27 and 28, write an expression for the area of the region in the figure.
27.
28.
**Number Problems** In Exercises 29–34, write a mathematical model for the problem and solve.

29. The sum of two consecutive natural numbers is 525. Find the numbers.
30. The sum of three consecutive natural numbers is 804. Find the numbers.
31. One positive number is 5 times another number. The difference between the two numbers is 148. Find the numbers.
32. One positive number is \( \frac{1}{3} \) of another number. The difference between the two numbers is 76. Find the numbers.
33. Find two consecutive integers whose product is 5 less than the square of the smaller number.
34. Find two consecutive natural numbers such that the difference of their reciprocals is \( \frac{1}{3} \) the reciprocal of the smaller number.

In Exercises 35–40, solve the percent equation.

35. What is 30% of 45? 
36. What is 175% of 360?
37. 432 is what percent of 1600?
38. 459 is what percent of 340?
39. 12 is \( \frac{1}{3} \)% of what number?
40. 70 is 40% of what number?

41. **Finance** A family has annual loan payments equaling 58.6% of their annual income. During the year, their loan payments total $13,077.75. What is their annual income?

42. **Government** The tables show the sources of income (in billions of dollars) and expenses (in billions of dollars) for the federal government in 2002. (Source: U.S. Office of Management and Budget)

<table>
<thead>
<tr>
<th>Source of income</th>
<th>Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corporation taxes</td>
<td>148.0</td>
</tr>
<tr>
<td>Income tax</td>
<td>858.3</td>
</tr>
<tr>
<td>Social Security</td>
<td>700.8</td>
</tr>
<tr>
<td>Other</td>
<td>146.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source of expenses</th>
<th>Expense</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest on debt</td>
<td>171.0</td>
</tr>
<tr>
<td>Health and human services</td>
<td>1317.9</td>
</tr>
<tr>
<td>Defense department</td>
<td>348.6</td>
</tr>
<tr>
<td>Other</td>
<td>173.5</td>
</tr>
</tbody>
</table>

(c) Compare the total income and total expenses. How much of a surplus or deficit is there?

In Exercises 43–46, the prices of various items are given for 1990 and 2002. Find the percent change for each item. (Source: 2003 Statistical Abstract of the U.S.; IDC; Consumer Electronics Association)

<table>
<thead>
<tr>
<th>Item</th>
<th>1990</th>
<th>2002</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gallon of regular unleaded gasoline</td>
<td>$1.16</td>
<td>$1.36</td>
</tr>
<tr>
<td>Half-gallon of ice cream</td>
<td>$2.54</td>
<td>$3.76</td>
</tr>
<tr>
<td>Pound of tomatoes</td>
<td>$0.86</td>
<td>$1.66</td>
</tr>
<tr>
<td>Personal computer</td>
<td>$1050</td>
<td>$855</td>
</tr>
</tbody>
</table>

47. **Discount** The price of a swimming pool has been discounted 16.5%. The sale price is $1210.75. Find the original list price of the pool.

48. **Finance** A salesperson’s weekly paycheck is 15% less than her coworker’s paycheck. The two paychecks total $645. Find the amount of each paycheck.
49. **Dimensions of a Room** A room is 1.5 times as long as it is wide, and its perimeter is 25 meters.
   
   (a) Draw a diagram that represents the problem. Identify the length as \( l \) and the width as \( w \).
   
   (b) Write \( l \) in terms of \( w \) and write an equation for the perimeter in terms of \( w \).
   
   (c) Find the dimensions of the room.

50. **Dimensions of a Picture Frame** A picture frame has a total perimeter of 2 meters. The height of the frame is 0.62 times its width.
   
   (a) Draw a diagram that represents the problem. Identify the width as \( w \) and the height as \( h \).
   
   (b) Write \( h \) in terms of \( w \) and write an equation for the perimeter in terms of \( w \).
   
   (c) Find the dimensions of the picture frame.

51. **Course Grade** To get an A in a course, you must have an average of at least 90 on four tests of 100 points each. The scores on your first three tests were 87, 92, and 84. What must you score on the fourth test to get an A for the course?

52. **Course Grade** You are taking a course that has four tests. The first three tests are 100 points each and the fourth test is 200 points. To get an A in the course, you must have an average of at least 90% on the four tests. Your scores on the first three tests were 87, 92, and 84. What must you score on the fourth test to get an A for the course?

53. **Travel Time** You are driving on a Canadian freeway to a town that is 300 kilometers from your home. After 30 minutes you pass a freeway exit that you know is 50 kilometers from your home. Assuming that you continue at the same constant speed, how long will it take for the entire trip?

54. **Travel Time** Two cars start at an interstate interchange and travel in the same direction at average speeds of 40 miles per hour and 55 miles per hour. How much time must elapse before the two cars are 5 miles apart?

55. **Travel Time** On the first part of a 317-mile trip, a salesperson averaged 58 miles per hour. He averaged only 52 miles per hour on the last part of the trip because of an increased volume of traffic. The total time of the trip was 5 hours and 45 minutes. Find the amount of time at each of the two speeds.

56. **Travel Time** Students are traveling in two cars to a football game 135 miles away. The first car leaves on time and travels at an average speed of 45 miles per hour. The second car starts \( \frac{1}{2} \) hour later and travels at an average speed of 55 miles per hour. How long will it take the second car to catch up to the first car? Will the second car catch up to the first car before the first car arrives at the game?

57. **Travel Time** Two families meet at a park for a picnic. At the end of the day, one family travels east at an average speed of 42 miles per hour and the other travels west at an average speed of 50 miles per hour. Both families have approximately 160 miles to travel.
   
   (a) Find the time it takes each family to get home.
   
   (b) Find the time that will have elapsed when they are 100 miles apart.
   
   (c) Find the distance the eastbound family has to travel after the westbound family has arrived home.

58. **Average Speed** A truck driver traveled at an average speed of 55 miles per hour on a 200-mile trip to pick up a load of freight. On the return trip (with the truck fully loaded), the average speed was 40 miles per hour. What was the average speed for the round trip?

59. **Wind Speed** An executive flew in the corporate jet to a meeting in a city 1500 kilometers away. After traveling the same amount of time on the return flight, the pilot mentioned that they still had 300 kilometers to go. The air speed of the plane was 600 kilometers per hour. How fast was the wind blowing? (Assume that the wind direction was parallel to the flight path and constant all day.)

60. **Physics** Light travels at the speed of approximately \( 3.0 \times 10^8 \) meters per second. Find the time in minutes required for light to travel from the sun to Earth (an approximate distance of \( 1.5 \times 10^{11} \) meters).

61. **Radio Waves** Radio waves travel at the same speed as light, approximately \( 3.0 \times 10^8 \) meters per second. Find the time required for a radio wave to travel from Mission Control in Houston to NASA astronauts on the surface of the moon \( 3.84 \times 10^8 \) meters away.

62. **Height of a Tree** To obtain the height of a tree (see figure), you measure the tree’s shadow and find that it is 8 meters long. You also measure the shadow of a two-meter lamppost and find that it is 75 centimeters long. How tall is the tree?

63. **Height of a Building** To obtain the height of the Chrysler Building in New York, you measure the building’s shadow and find that it is 87 feet long. You also measure the shadow of a four-foot stake and find that it is 4 inches long. How tall is the Chrysler Building?
64. Flagpole Height A person who is 6 feet tall walks away from a flagpole toward the tip of the shadow of the flagpole. When the person is 30 feet from the flagpole, the tips of the person’s shadow and the shadow cast by the flagpole coincide at a point 5 feet in front of the person.

(a) Draw a diagram that gives a visual representation of the problem. Let \( h \) represent the height of the flagpole.

(b) Find the height of the flagpole.

65. Shadow Length A person who is 6 feet tall walks away from a 50-foot silo toward the tip of the silo’s shadow. At a distance of 32 feet from the silo, the person’s shadow begins to emerge beyond the silo’s shadow. How much farther must the person walk to be completely out of the silo’s shadow?

66. Investment You plan to invest $12,000 in two funds paying \( 4\frac{1}{2}\% \) and 5% simple interest. (There is more risk in the 5% fund.) Your goal is to obtain a total annual interest income of $380 from the investments. What is the smallest amount you can invest in the 5% fund and still meet your objective?

67. Investment You plan to invest $25,000 in two funds paying 3% and \( 4\frac{1}{2}\% \) simple interest. (There is more risk in the \( 4\frac{1}{2}\% \) fund.) Your goal is to obtain a total annual interest income of $1000 from the investments. What is the smallest amount you can invest in the \( 4\frac{1}{2}\% \) fund and still meet your objective?

68. Inventory A nursery has $20,000 of inventory in dogwood trees and red maple trees. The profit on a dogwood tree is 25% and the profit on a red maple tree is 17%. The profit for the entire stock is 20%. How much was invested in each type of tree?

69. Inventory An automobile dealer has $600,000 of inventory in minivans and SUVs. The profit on a minivan is 24% and the profit on an SUV is 28%. The profit for the entire stock is 25%. How much was invested in each type of vehicle?

70. Mixture Problem Using the values in the table, determine the amounts of solutions 1 and 2, needed to obtain the desired amount and concentration of the final mixture.

<table>
<thead>
<tr>
<th></th>
<th>Concentration</th>
<th>Amount of final solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Solution 1</td>
<td>Solution 2</td>
</tr>
<tr>
<td>(a)</td>
<td>10%</td>
<td>30%</td>
</tr>
<tr>
<td>(b)</td>
<td>25%</td>
<td>50%</td>
</tr>
<tr>
<td>(c)</td>
<td>15%</td>
<td>45%</td>
</tr>
<tr>
<td>(d)</td>
<td>70%</td>
<td>90%</td>
</tr>
</tbody>
</table>

71. Mixture Problem A 100% concentrate is to be mixed with a mixture having a concentration of 40% to obtain 55 gallons of a mixture with a concentration of 75%. How much of the 100% concentrate will be needed?

72. Mixture Problem A forester mixes gasoline and oil to make 2 gallons of mixture for his two-cycle chain-saw engine. This mixture is 32 parts gasoline and 1 part two-cycle oil. How much gasoline must be added to bring the mixture to 40 parts gasoline and 1 part oil?

73. Mixture Problem A grocer mixes peanuts that cost $2.49 per pound and walnuts that cost $3.89 per pound to make 100 pounds of a mixture that costs $3.19 per pound. How much of each kind of nut is put into the mixture?

74. Company Costs An outdoor furniture manufacturer has fixed costs of $10,000 per month and average variable costs of $8.50 per unit manufactured. The company has $85,000 available to cover the monthly costs. How many units can the company manufacture? (Fixed costs are those that occur regardless of the level of production. Variable costs depend on the level of production.)

75. Company Costs A plumbing supply company has fixed costs of $10,000 per month and average variable costs of $9.30 per unit manufactured. The company has $85,000 available to cover the monthly costs. How many units can the company manufacture? (Fixed costs are those that occur regardless of the level of production. Variable costs depend on the level of production.)

76. Water Depth A trough is 12 feet long, 3 feet deep, and 3 feet wide (see figure). Find the depth of the water when the trough contains 70 gallons (1 gallon = 0.13368 cubic foot).

In Exercises 77–88, solve for the indicated variable.

77. Area of a Triangle
Solve for \( h \): \( A = \frac{1}{2}bh \)

78. Area of a Trapezoid
Solve for \( b \): \( A = \frac{1}{2}(a + b)h \)

79. Markup
Solve for \( C \): \( S = C + RC \)

80. Investment at Simple Interest
Solve for \( r \): \( A = P + Prt \)
81. **Volume of an Oblate Spheroid**
   Solve for $b$: $V = \frac{1}{2} \pi a^2 b$

82. **Volume of a Spherical Segment**
   Solve for $r$: $V = \frac{1}{2} \pi h^2 (3r - h)$

83. **Free-Falling Body**
   Solve for $a$: $h = v_0 t + \frac{1}{2} a t^2$

84. **Lensmaker’s Equation**
   Solve for $R_1$: $\frac{1}{f} = \left( n - 1 \right) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$

85. **Capacitance in Series Circuits**
   Solve for $C_1$: $C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$

86. **Arithmetic Progression**
   Solve for $a$: $S = \frac{n}{2} [2a + (n - 1)d]$

87. **Arithmetic Progression**
   Solve for $n$: $L = a + (n - 1)d$

88. **Geometric Progression**
   Solve for $r$: $S = \frac{rL - a}{r - 1}$

**Physics**

In Exercises 89 and 90, you have a uniform beam of length $L$ with a fulcrum $x$ feet from one end (see figure). Objects with weights $W_1$ and $W_2$ are placed at opposite ends of the beam. The beam will balance when $W_1 x = W_2 (L - x)$. Find $x$ such that the beam will balance.

![Diagram of a beam with weights and fulcrum](image)

89. Two children weighing 50 pounds and 75 pounds are playing on a seesaw that is 10 feet long.

90. A person weighing 200 pounds is attempting to move a 550-pound rock with a bar that is 5 feet long.

91. **Volume of a Billiard Ball**
   A billiard ball has a volume of 5.96 cubic inches. Find the radius of a billiard ball.

92. **Length of a Tank**
   The diameter of a cylindrical propane gas tank is 4 feet. The total volume of the tank is 603.2 cubic feet. Find the length of the tank.

93. **Temperature**
   The average daily temperature in San Diego, California is 64.4°F. What is San Diego’s average daily temperature in degrees Celsius? (Source: NOAA)

94. **Temperature**
   The average daily temperature in Duluth, Minnesota is 39.1°F. What is Duluth’s average daily temperature in degrees Celsius? (Source: NOAA)

95. **Temperature**
   The highest temperature ever recorded in Phoenix, Arizona was 50°C. What is this temperature in degrees Fahrenheit? (Source: NOAA)

96. **Temperature**
   The lowest temperature ever recorded in Louisville, Kentucky was −30°C. What is this temperature in degrees Fahrenheit? (Source: NOAA)

**Synthesis**

**True or False?** In Exercises 97 and 98, determine whether the statement is true or false. Justify your answer.

97. “8 less than $z$ cubed divided by the difference of $z$ squared and 9” can be written as
   \[ \frac{z^3 - 8}{(z^2 - 9)^2}. \]

98. The volume of a cube with a side of length 9.5 inches is greater than the volume of a sphere with a radius of 5.9 inches.

99. Consider the linear equation $ax + b = 0$.
   (a) What is the sign of the solution if $ab > 0$?
   (b) What is the sign of the solution if $ab < 0$?
   In each case, explain your reasoning.

100. Write a linear equation that has the solution $x = -3$. (There are many correct answers.)

**Skills Review**

In Exercises 101–104, simplify the expression.

101. $(5x^4)(25x^2)^{-1}$, $x \neq 0$

102. $\sqrt[3]{150x^4}^3$

103. $\frac{3}{x - 5} + \frac{2}{5 - x}$

104. $\frac{5}{x} + \frac{3x}{x^2 - 9} - \frac{10}{x + 3}$

In Exercises 105–108, rationalize the denominator.

105. $\frac{10}{7\sqrt{3}}$

106. $\frac{4}{3\sqrt{6}}$

107. $\frac{5}{\sqrt{6} + \sqrt{11}}$

108. $\frac{14}{3\sqrt{10} - 1}$
Factoring

A quadratic equation in \( x \) is an equation that can be written in the general form
\[
ax^2 + bx + c = 0
\]
where \( a, b, \) and \( c \) are real numbers with \( a \neq 0 \). A quadratic equation in \( x \) is also called a second-degree polynomial equation in \( x \).

In this section, you will study four methods for solving quadratic equations: factoring, extracting square roots, completing the square, and the Quadratic Formula. The first method is based on the Zero-Factor Property from Section P.1.

If \( ab = 0 \), then \( a = 0 \) or \( b = 0 \). Zero-Factor Property

To use this property, write the left side of the general form of a quadratic equation as the product of two linear factors. Then find the solutions of the quadratic equation by setting each linear factor equal to zero.

**Example 1**  
**Solving a Quadratic Equation by Factoring**

**a.**  
\[
2x^2 + 9x + 7 = 3
\]
\[
2x^2 + 9x + 4 = 0
\]
\[
(2x + 1)(x + 4) = 0
\]
\[
2x + 1 = 0 \quad \Rightarrow \quad x = -\frac{1}{2}
\]
\[
x + 4 = 0 \quad \Rightarrow \quad x = -4
\]
The solutions are \( x = -\frac{1}{2} \) and \( x = -4 \). Check these in the original equation.

**b.**  
\[
6x^2 - 3x = 0
\]
\[
3x(2x - 1) = 0
\]
\[
3x = 0 \quad \Rightarrow \quad x = 0
\]
\[
2x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2}
\]
The solutions are \( x = 0 \) and \( x = \frac{1}{2} \). Check these in the original equation.

**Checkpoint**  
Now try Exercise 9.

Be sure you see that the Zero-Factor Property works only for equations written in general form (in which the right side of the equation is zero). So, all terms must be collected on one side before factoring. For instance, in the equation \((x - 5)(x + 2) = 8\), it is incorrect to set each factor equal to 8. To solve this equation, you must multiply the binomials on the left side of the equation, and then subtract 8 from each side. After simplifying the left side of the equation, you can use the Zero-Factor Property to solve the equation. Try to solve this equation correctly.
Extracting Square Roots

There is a nice shortcut for solving quadratic equations of the form \( u^2 = d \), where \( d > 0 \) and \( u \) is an algebraic expression. By factoring, you can see that this equation has two solutions.

\[
\begin{align*}
\text{Write original equation.} & \quad u^2 = d \\
\text{Write in general form.} & \quad u^2 - d = 0 \\
\text{Factor.} & \quad (u + \sqrt{d})(u - \sqrt{d}) = 0 \\
\text{Set 1st factor equal to 0.} & \quad u + \sqrt{d} = 0 \quad \Rightarrow \quad u = -\sqrt{d} \\
\text{Set 2nd factor equal to 0.} & \quad u - \sqrt{d} = 0 \quad \Rightarrow \quad u = \sqrt{d}
\end{align*}
\]

Because the two solutions differ only in sign, you can write the solutions together, using a “plus or minus sign,” as

\[ u = \pm \sqrt{d}. \]

This form of the solution is read as “\( u \) is equal to plus or minus the square root of \( d \).” Solving an equation of the form \( u^2 = d \) without going through the steps of factoring is called **extracting square roots**.

Example 2

**Extracting Square Roots**

Solve each equation by extracting square roots.

a. \( 4x^2 = 12 \)  
   b. \( (x - 3)^2 = 7 \)

**Solution**

a. \( 4x^2 = 12 \)

\[
\begin{align*}
\text{Write original equation.} & \quad x^2 = 3 \\
\text{Divide each side by 4.} & \quad x = \pm \sqrt{3} \\
\text{Extract square roots.}
\end{align*}
\]

When you take the square root of a variable expression, you must account for both positive and negative solutions. So, the solutions are \( x = \sqrt{3} \) and \( x = -\sqrt{3} \). Check these in the original equation.

b. \( (x - 3)^2 = 7 \)

\[
\begin{align*}
\text{Write original equation.} & \quad x - 3 = \pm \sqrt{7} \\
\text{Extract square roots.} & \quad x = 3 \pm \sqrt{7} \\
\text{Add 3 to each side.}
\end{align*}
\]

The solutions are \( x = 3 \pm \sqrt{7} \). Check these in the original equation.

**Checkpoint**

Now try Exercise 29.
Completing the Square

The equation in Example 2(b) was given in the form \((x - 3)^2 = 7\) so that you could find the solution by extracting square roots. Suppose, however, that the equation had been given in the general form \(x^2 - 6x + 2 = 0\). Because this equation is equivalent to the original, it has the same two solutions, \(x = 3 \pm \sqrt{7}\). However, the left side of the equation is not factorable, and you cannot find its solutions unless you rewrite the equation by completing the square. Note that when you complete the square to solve a quadratic equation, you are just rewriting the equation so it can be solved by extracting square roots.

Completing the Square: Leading Coefficient Is 1

Solve \(x^2 + 2x - 6 = 0\) by completing the square.

Solution

\[
\begin{align*}
\text{Write original equation.} \\
x^2 + 2x - 6 = 0 & & \text{Add 6 to each side.} \\
x^2 + 2x = 6 & & \text{Add } 1^2 \text{ to each side.} \\
(x + 1)^2 = 7 & & \text{(half of 2)}^2 \text{ Simplify.} \\
x + 1 = \pm \sqrt{7} & & \text{Take square root of each side.} \\
x = -1 \pm \sqrt{7} & & \text{Subtract 1 from each side.}
\end{align*}
\]

The solutions are \(x = -1 \pm \sqrt{7}\). Check these in the original equation as follows.

Check

\[
\begin{align*}
\text{Write original equation.} \\
x^2 + 2x - 6 = 0 & & \text{Substitute } -1 + \sqrt{7} \text{ for } x. \\
(-1 + \sqrt{7})^2 + 2(-1 + \sqrt{7}) - 6 \stackrel{?}{=} 0 & & \text{Multiply.} \\
8 - 2\sqrt{7} - 2 + 2\sqrt{7} - 6 \stackrel{?}{=} 0 & & \text{Solution checks } ✔
\end{align*}
\]

Check the second solution in the original equation.

\[
\begin{align*}
\text{Now try Exercise 37.}
\end{align*}
\]

When solving quadratic equations by completing the square, you must add \((b/2)^2\) to each side in order to maintain equality. If the leading coefficient is not 1, you must divide each side of the equation by the leading coefficient before completing the square, as shown in Example 4.
**Example 4  Completing the Square: Leading Coefficient Is Not 1**

Solve $2x^2 + 8x + 3 = 0$ by completing the square.

**Solution**

Write original equation.

Simplify.

Rationalize denominator.

Take square root of each side.

Subtract 2 from each side.

The solutions are $x = -2 \pm \frac{\sqrt{10}}{2}$. Check these in the original equation.

**Example 5  Completing the Square: Leading Coefficient Is Not 1**

$3x^2 - 4x - 5 = 0$

Add 5 to each side.

Divide each side by 3.

Add $\left(-\frac{2}{3}\right)^2$ to each side.

Simplify.

Perfect square trinomial.

Extract square roots.

Solutions
The Quadratic Formula

Often in mathematics you are taught the long way of solving a problem first. Then, the longer method is used to develop shorter techniques. The long way stresses understanding and the short way stresses efficiency.

For instance, you can think of completing the square as a “long way” of solving a quadratic equation. When you use completing the square to solve quadratic equations, you must complete the square for each equation separately. In the following derivation, you complete the square once in a general setting to obtain the Quadratic Formula—a shortcut for solving quadratic equations.

Write in general form, \( a \neq 0 \).

\[ ax^2 + bx + c = 0 \]
Write in general form, \( a \neq 0 \).

\[ ax^2 + bx = -c \]
Subtract \( c \) from each side.

\[ x^2 + \frac{b}{a} x = -\frac{c}{a} \]
Divide each side by \( a \).

\[ x^2 + \frac{b}{a} x + \left( \frac{b}{2a} \right)^2 = -\frac{c}{a} + \left( \frac{b}{2a} \right)^2 \]
Complete the square.

\[ \left( \frac{b}{2a} \right)^2 \]

\[ \left( x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2} \]
Simplify.

\[ x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \]
Extract square roots.

\[ x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2|a|} \]
Solutions

Note that because \( \pm 2|a| \) represents the same numbers as \( \pm 2a \), you can omit the absolute value sign. So, the formula simplifies to

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]

The Quadratic Formula

The solutions of a quadratic equation in the general form

\[ ax^2 + bx + c = 0, \quad a \neq 0 \]

are given by the Quadratic Formula

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]

The Quadratic Formula is one of the most important formulas in algebra. You should learn the verbal statement of the Quadratic Formula:

“Negative \( b \), plus or minus the square root of \( b \) squared minus 4\( ac \), all divided by \( 2a \).”
From each graph, can you tell whether the discriminant is positive, zero, or negative? Explain your reasoning. Find each discriminant to verify your answers.

a. \(x^2 - 2x = 0\)

b. \(x^2 - 2x + 1 = 0\)

c. \(x^2 - 2x + 2 = 0\)

How many solutions would part (c) have if the linear term was 2x? If the constant was -2?

In the Quadratic Formula, the quantity under the radical sign, \(b^2 - 4ac\), is called the **discriminant** of the quadratic expression \(ax^2 + bx + c\). It can be used to determine the nature of the solutions of a quadratic equation.

**Solutions of a Quadratic Equation**

The solutions of a quadratic equation \(ax^2 + bx + c = 0, a \neq 0\), can be classified as follows. If the discriminant \(b^2 - 4ac\) is

1. **positive**, then the quadratic equation has **two** distinct real solutions and its graph has **two** x-intercepts.
2. **zero**, then the quadratic equation has **one** repeated real solution and its graph has **one** x-intercept.
3. **negative**, then the quadratic equation has **no** real solutions and its graph has **no** x-intercepts.

If the discriminant of a quadratic equation is negative, as in case 3 above, then its square root is imaginary (not a real number) and the Quadratic Formula yields two complex solutions. You will study complex solutions in Section 1.5.

When using the Quadratic Formula, remember that **before** the formula can be applied, you must first write the quadratic equation in general form.

**Example 6 The Quadratic Formula: Two Distinct Solutions**

Use the Quadratic Formula to solve \(x^2 + 3x = 9\).

**Solution**

The general form of the equation is \(x^2 + 3x - 9 = 0\). The discriminant is \(b^2 - 4ac = 9 + 36 = 45\), which is positive. So, the equation has two real solutions. You can solve the equation as follows.

\[
x^2 + 3x - 9 = 0
\]

Write in general form.

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

Quadratic Formula

\[
x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-9)}}{2(1)}
\]

Substitute \(a = 1, b = 3,\) and \(c = -9\).

\[
x = \frac{-3 \pm \sqrt{45}}{2}
\]

Simplify.

\[
x = \frac{-3 \pm 3\sqrt{5}}{2}
\]

Simplify.

The two solutions are:

\[
x = \frac{-3 + 3\sqrt{5}}{2} \quad \text{and} \quad x = \frac{-3 - 3\sqrt{5}}{2}.
\]

Check these in the original equation.

Now try Exercise 75.
Applications

Quadratic equations often occur in problems dealing with area. Here is a simple example. “A square room has an area of 144 square feet. Find the dimensions of the room.” To solve this problem, let \( x \) represent the length of each side of the room. Then, by solving the equation

\[ x^2 = 144 \]

you can conclude that each side of the room is 12 feet long. Note that although the equation \( x^2 = 144 \) has two solutions, \( x = -12 \) and \( x = 12 \), the negative solution does not make sense in the context of the problem, so you choose the positive solution.

**Example 7** Finding the Dimensions of a Room

A bedroom is 3 feet longer than it is wide (see Figure 1.19) and has an area of 154 square feet. Find the dimensions of the room.

![Figure 1.19](image)

**Solution**

**Verbal Model:** Width of room \( \times \) Length of room = Area of room

**Labels:**
- Width of room = \( w \) (feet)
- Length of room = \( w + 3 \) (feet)
- Area of room = 154 (square feet)

**Equation:**

\[ w(w + 3) = 154 \]
\[ w^2 + 3w - 154 = 0 \]
\[ (w - 11)(w + 14) = 0 \]

\[ w - 11 = 0 \quad \Rightarrow \quad w = 11 \]
\[ w + 14 = 0 \quad \Rightarrow \quad w = -14 \]

Choosing the positive value, you find that the width is 11 feet and the length is \( w + 3 \), or 14 feet. You can check this solution by observing that the length is 3 feet longer than the width and that the product of the length and width is 154 square feet.

**Checkpoint** Now try Exercise 109.
Another common application of quadratic equations involves an object that is falling (or projected into the air). The general equation that gives the height of such an object is called a **position equation**, and on Earth’s surface it has the form

\[ s = -16t^2 + v_0 t + s_0. \]

In this equation, \( s \) represents the height of the object (in feet), \( v_0 \) represents the initial velocity of the object (in feet per second), \( s_0 \) represents the initial height of the object (in feet), and \( t \) represents the time (in seconds).

**Example 8  Falling Time**

A construction worker on the 24th floor of a building project (see Figure 1.20) accidentally drops a wrench and yells “Look out below!” Could a person at ground level hear this warning in time to get out of the way? *(Note: The speed of sound is about 1100 feet per second.)*

**Solution**

Assume that each floor of the building is 10 feet high, so that the wrench is dropped from a height of 235 feet (the construction worker’s hand is 5 feet below the ceiling of the 24th floor). Because sound travels at about 1100 feet per second, it follows that a person at ground level hears the warning within 1 second of the time the wrench is dropped. To set up a mathematical model for the height of the wrench, use the position equation

\[ s = -16t^2 + v_0 t + s_0. \]

Because the object is dropped rather than thrown, the initial velocity is \( v_0 = 0 \) feet per second. Moreover, because the initial height is \( s_0 = 235 \) feet, you have the following model.

\[ s = -16t^2 + 0t + 235 = -16t^2 + 235 \]

After the wrench has fallen for 1 second, its height is \(-16(1)^2 + 235 = 219\) feet. After the wrench has fallen for 2 seconds, its height is \(-16(2)^2 + 235 = 171\) feet. To find the number of seconds it takes the wrench to hit the ground, let the height \( s \) be zero and solve the equation for \( t \).

\[ s = -16t^2 + 235 \quad \text{Write position equation.} \]
\[ 0 = -16t^2 + 235 \quad \text{Substitute 0 for height.} \]
\[ 16t^2 = 235 \quad \text{Add } 16t^2 \text{ to each side.} \]
\[ t^2 = \frac{235}{16} \quad \text{Divide each side by } 16. \]
\[ t = \frac{\sqrt{235}}{4} \quad \text{Extract positive square root.} \]
\[ t \approx 3.83 \quad \text{Use a calculator.} \]

The wrench will take about 3.83 seconds to hit the ground. If the person hears the warning 1 second after the wrench is dropped, the person still has almost 3 seconds to get out of the way.

**STUDY TIP**

The position equation used in Example 8 ignores air resistance. This implies that it is appropriate to use the position equation only to model falling objects that have little air resistance and that fall over short distances.

**CHECKPOINT**

Now try Exercise 115.
A third type of application of a quadratic equation is one in which a quantity is changing over time $t$ according to a quadratic model.

**Example 9  Quadratic Modeling: Internet Users**

From 2000 to 2007, the estimated number of Internet users $I$ (in millions) in the United States can be modeled by the quadratic equation

$$I = -1.163t^2 + 17.19t + 125.9, \quad 0 \leq t \leq 7$$

where $t$ represents the year, with $t = 0$ corresponding to 2000. According to this model, in which year will the number of Internet users reach or surpass 180 million? (Source: eMarketer)

**Algebraic Solution**

To find the year in which the number of Internet users will reach 180 million, you need to solve the equation

$$-1.163t^2 + 17.19t + 125.9 = 180.$$  

To begin, write the equation in general form.

$$-1.163t^2 + 17.19t - 54.1 = 0$$

Then apply the Quadratic Formula.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-17.19 \pm \sqrt{17.19^2 - 4(-1.163)(-54.1)}}{2(-1.163)}$$

$$t \approx \frac{-17.19 \pm \sqrt{43.82}}{-2.326}$$

$$t \approx 4.5 \text{ or } 10.2$$

Choose the smaller value $t \approx 4.5$. Because $t = 0$ corresponds to 2000, it follows that $t \approx 4.5$ must correspond to 2004. So, the number of Internet users should have reached 180 million during the year 2004.

**Numerical Solution**

You can estimate the year in which the number of Internet users will reach or surpass 180 million by constructing a table of values. The table below shows the number of Internet users for each year from 2000 to 2007.

<table>
<thead>
<tr>
<th>Year</th>
<th>$t$</th>
<th>$I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>0</td>
<td>125.9</td>
</tr>
<tr>
<td>2001</td>
<td>1</td>
<td>141.9</td>
</tr>
<tr>
<td>2002</td>
<td>2</td>
<td>155.6</td>
</tr>
<tr>
<td>2003</td>
<td>3</td>
<td>167.0</td>
</tr>
<tr>
<td>2004</td>
<td>4</td>
<td>176.1</td>
</tr>
<tr>
<td>2005</td>
<td>5</td>
<td>182.8</td>
</tr>
<tr>
<td>2006</td>
<td>6</td>
<td>187.2</td>
</tr>
<tr>
<td>2007</td>
<td>7</td>
<td>189.2</td>
</tr>
</tbody>
</table>

From the table, you can see that sometime during 2004 the number of Internet users reached 180 million.

**Technology**

You can also use a graphical approach to solve Example 9. Use a graphing utility to graph

$$y_1 = -1.163t^2 + 17.19t + 125.9 \quad \text{and} \quad y_2 = 180$$

in the same viewing window. Then use the intersect feature to find the point(s) of intersection of the two graphs. You should obtain $t \approx 4.5$, which verifies the answer obtained algebraically.
A fourth type of application that often involves a quadratic equation is one dealing with the hypotenuse of a right triangle. In these types of applications, the **Pythagorean Theorem** is often used. The Pythagorean Theorem states that

\[
  a^2 + b^2 = c^2
\]

where \(a\) and \(b\) are the legs of a right triangle and \(c\) is the hypotenuse.

### Example 10  An Application Involving the Pythagorean Theorem

An L-shaped sidewalk from the athletic center to the library on a college campus is shown in Figure 1.21. The sidewalk was constructed so that the length of one sidewalk forming the L was twice as long as the other. The length of the diagonal sidewalk that cuts across the grounds between the two buildings is 32 feet. How many feet does a person save by walking on the diagonal sidewalk?

**Solution**

Using the Pythagorean Theorem, you have the following.

\[
  x^2 + (2x)^2 = 32^2
\]

Combine like terms.

\[
  5x^2 = 1024
\]

Divide each side by 5.

\[
  x^2 = 204.8
\]

Take the square root of each side.

\[
  x = \pm \sqrt{204.8}
\]

Extract positive square root.

\[
  x = \sqrt{204.8} = 204.8
\]

The total distance covered by walking on the L-shaped sidewalk is

\[
  x + 2x = 3x
\]

\[
  = 3 \times \sqrt{204.8}
\]

\[
  \approx 42.9 \text{ feet.}
\]

Walking on the diagonal sidewalk saves a person about \(42.9 - 32 = 10.9\) feet.

**Checkpoint**  Now try Exercise 123.

---

**Writing About Mathematics**

**Comparing Solution Methods**  In this section, you studied four algebraic methods for solving quadratic equations. Solve each of the quadratic equations below in several different ways. Write a short paragraph explaining which method(s) you prefer. Does your preferred method depend on the equation?

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>(x^2 - 4x - 5 = 0)</td>
</tr>
<tr>
<td>b.</td>
<td>(x^2 - 4x = 0)</td>
</tr>
<tr>
<td>c.</td>
<td>(x^2 - 4x - 3 = 0)</td>
</tr>
<tr>
<td>d.</td>
<td>(x^2 - 4x - 6 = 0)</td>
</tr>
</tbody>
</table>
1.4 Exercises

VOCABULARY CHECK: Fill in the blanks.
1. An equation of the form \( ax^2 + bx + c = 0, a \neq 0 \) is a ________ ________, or second-degree polynomial equation in \( x \).
2. The four methods that can be used to solve a quadratic equation are ________, ________, ________, and ________.
3. The part of the Quadratic Formula, \( b^2 - 4ac \), known as the ________, determines the type of solutions of a quadratic equation.
4. The general equation that gives the height of an object (in feet) in terms of the time (in seconds) is called the ________ equation, and has the form \( s = ________ \), where \( v_0 \) represents the ________ and \( s_0 \) represents the ________.
5. An important theorem that is sometimes used in applications that require solving quadratic equations is the ________ ________.


In Exercises 1–6, write the quadratic equation in general form.
1. \( 2x^2 = 3 - 8x \)
2. \( x^2 = 16x \)
3. \( (x - 3)^2 = 3 \)
4. \( 13 - 3(x + 7)^2 = 0 \)
5. \( \frac{1}{3}(3x^2 - 10) = 18x \)
6. \( x(x + 2) = 5x^2 + 1 \)

In Exercises 7–20, solve the quadratic equation by factoring.
7. \( 6x^2 + 3x = 0 \)
8. \( 9x^2 - 1 = 0 \)
9. \( x^2 - 2x - 8 = 0 \)
10. \( x^2 - 10x + 9 = 0 \)
11. \( x^2 + 10x + 25 = 0 \)
12. \( 4x^2 + 12x + 9 = 0 \)
13. \( 3 + 5x - 2x^2 = 0 \)
14. \( 2x^2 = 19x + 33 \)
15. \( x^2 + 4x = 12 \)
16. \( -x^2 + 8x = 12 \)
17. \( \frac{3}{2}x^2 + 8x + 20 = 0 \)
18. \( \frac{1}{8}x^2 - x - 16 = 0 \)
19. \( x^2 + 2ax + a^2 = 0, a \) is a real number
20. \( (x + a)^2 - b^2 = 0, a \) and \( b \) are real numbers

In Exercises 21–34, solve the equation by extracting square roots.
21. \( x^2 = 49 \)
22. \( x^2 = 169 \)
23. \( x^2 = 11 \)
24. \( x^2 = 32 \)
25. \( 3x^2 = 81 \)
26. \( 9x^2 = 36 \)
27. \( (x - 12)^2 = 16 \)
28. \( (x + 13)^2 = 25 \)
29. \( (x + 2)^2 = 14 \)
30. \( (x - 5)^2 = 30 \)
31. \( (2x - 1)^2 = 18 \)
32. \( (4x + 7)^2 = 44 \)
33. \( (x - 7)^2 = (x + 3)^2 \)
34. \( (x + 5)^2 = (x + 4)^2 \)

In Exercises 35–44, solve the quadratic equation by completing the square.
35. \( x^2 + 4x - 32 = 0 \)
36. \( x^2 - 2x - 3 = 0 \)
37. \( x^2 + 12x + 25 = 0 \)
38. \( x^2 + 8x + 14 = 0 \)
39. \( 9x^2 - 18x = -3 \)
40. \( 9x^2 - 12x = 14 \)
41. \( 8 + 4x - x^2 = 0 \)
42. \( -x^2 + x - 1 = 0 \)
43. \( 2x^2 + 5x - 8 = 0 \)
44. \( 4x^2 - 4x - 99 = 0 \)

In Exercises 45–50, rewrite the quadratic portion of the algebraic expression as the sum or difference of two squares by completing the square.
45. \( \frac{1}{x^2 + 2x + 5} \)
46. \( \frac{1}{x^2 - 12x + 19} \)
47. \( \frac{4}{x^2 + 4x - 3} \)
48. \( \frac{5}{x^2 + 25x + 11} \)
49. \( \frac{1}{\sqrt{6x - x^2}} \)
50. \( \frac{1}{\sqrt{16 - 6x - x^2}} \)
In Exercises 51–58, (a) use a graphing utility to graph the equation, (b) use the graph to approximate any x-intercepts of the graph, (c) set y = 0 and solve the resulting equation, and (d) compare the result of part (c) with the x-intercepts of the graph.

51. \( y = (x + 3)^2 - 4 \)  
52. \( y = (x - 4)^2 - 1 \)  
53. \( y = 1 - (x - 2)^2 \)  
54. \( y = 9 - (x - 8)^2 \)  
55. \( y = -4x^2 + 4x + 3 \)  
56. \( y = 4x^2 - 1 \)  
57. \( y = x^2 + 3x - 4 \)  
58. \( y = x^2 - 5x - 24 \)

In Exercises 59–66, use the discriminant to determine the number of real solutions of the quadratic equation.

59. \( 2x^2 - 5x + 5 = 0 \)  
60. \( -5x^2 - 4x + 1 = 0 \)  
61. \( 2x^2 - x - 1 = 0 \)  
62. \( x^2 - 4x + 4 = 0 \)  
63. \( \frac{1}{3}x^2 - 5x + 25 = 0 \)  
64. \( \frac{3}{4}x^2 - 8x + 28 = 0 \)  
65. \( 0.2x^2 + 1.2x - 8 = 0 \)  
66. \( 9 + 2.4x - 8.3x^2 = 0 \)

In Exercises 67–90, use the Quadratic Formula to solve the equation.

67. \( 2x^2 + x - 1 = 0 \)  
68. \( 2x^2 - x - 1 = 0 \)  
69. \( 16x^2 + 8x - 3 = 0 \)  
70. \( 25x^2 - 20x + 3 = 0 \)  
71. \( 2 + 2x - x^2 = 0 \)  
72. \( x^2 - 10x + 22 = 0 \)  
73. \( x^2 + 14x + 44 = 0 \)  
74. \( 6x = 4 - x^2 \)  
75. \( x^2 + 8x - 4 = 0 \)  
76. \( 4x^2 - 4x - 4 = 0 \)  
77. \( 12x - 9x^2 = -3 \)  
78. \( 16x^2 + 22 = 40x \)  
79. \( 9x^2 + 24x + 16 = 0 \)  
80. \( 36x^2 + 24x - 7 = 0 \)  
81. \( 4x^2 + 4x = 7 \)  
82. \( 16x^2 - 40x + 5 = 0 \)  
83. \( 28x - 49x^2 = 4 \)  
84. \( 3x + x^2 - 1 = 0 \)  
85. \( 8r = 5 + 2r^2 \)  
86. \( 25h^2 + 80h + 61 = 0 \)  
87. \( (y - 5)^2 = 2y \)  
88. \( (z + 6)^2 = -2z \)  
89. \( \frac{1}{3}x^2 + \frac{3}{4}x = 2 \)  
90. \( \left(\frac{2}{3}x - 14\right)^2 = 8x \)

In Exercises 91–98, use the Quadratic Formula to solve the equation. (Round your answer to three decimal places.)

91. \( 5.1x^2 - 1.7x - 3.2 = 0 \)  
92. \( 2x^2 - 2.50x - 0.42 = 0 \)  
93. \( -0.067x^2 - 0.852x + 1.277 = 0 \)  
94. \( -0.005x^2 + 0.101x - 0.193 = 0 \)  
95. \( 422x^2 - 506x - 347 = 0 \)  
96. \( 1100x^2 + 326x - 715 = 0 \)  
97. \( 12.67x^2 + 31.55x + 8.09 = 0 \)  
98. \( -3.22x^2 - 0.08x + 28.651 = 0 \)

In Exercises 99–108, solve the equation using any convenient method.

99. \( x^2 - 2x - 1 = 0 \)
100. \( 11x^2 + 33x = 0 \)
101. \( (x + 3)^2 = 81 \)
102. \( x^2 - 14x + 49 = 0 \)
103. \( x^2 - x - \frac{11}{4} = 0 \)
104. \( x^2 + 3x - \frac{3}{4} = 0 \)
105. \( (x + 1)^2 = x^2 \)
106. \( a^2x^2 - b^2 = 0 \), \( a \) and \( b \) are real numbers
107. \( 3x + 4 = 2x^2 - 7 \)
108. \( 4x^2 + 2x + 4 = 2x + 8 \)

109. **Floor Space** The floor of a one-story building is 14 feet longer than it is wide. The building has 1632 square feet of floor space.

(a) Draw a diagram that gives a visual representation of the floor space. Represent the width as \( w \) and show the length in terms of \( w \).

(b) Write a quadratic equation in terms of \( w \).

(c) Find the length and width of the floor of the building.

110. **Dimensions of a Corral** A rancher has 100 meters of fencing to enclose two adjacent rectangular corrals (see figure). The rancher wants the enclosed area to be 350 square meters. What dimensions should the rancher use to obtain this area?
111. **Packaging** An open box with a square base (see figure) is to be constructed from 84 square inches of material. The height of the box is 2 inches. What are the dimensions of the box? (Hint: The surface area is $S = x^2 + 4xh$.)

112. **Packaging** An open gift box is to be made from a square piece of material by cutting two-centimeter squares from the corners and turning up the sides (see figure). The volume of the finished box is to be 200 cubic centimeters. Find the size of the original piece of material.

113. **Mowing the Lawn** Two landscapers must mow a rectangular lawn that measures 100 feet by 200 feet. Each wants to mow no more than half of the lawn. The first starts by mowing around the outside of the lawn. The mower has a 24-inch cut. How wide a strip must the first landscaper mow on each of the four sides in order to mow no more than half of the lawn? Approximate the required number of trips around the lawn the first landscaper must take.

114. **Seating** A rectangular classroom seats 72 students. If the seats were rearranged with three more seats in each row, the classroom would have two fewer rows. Find the original number of seats in each row.

In Exercises 115–118, use the position equation given in Example 8 as the model for the problem.

115. **Military** A C-141 Starlifter flying at 32,000 feet over level terrain drops a 500-pound supply package.
   (a) How long will it take until the supply package strikes the ground?
   (b) The plane is flying at 500 miles per hour. How far will the supply package travel horizontally during its descent?

116. **Eiffel Tower** You drop a coin from the top of the Eiffel Tower in Paris. The building has a height of 984 feet.
   (a) Use the position equation to write a mathematical model for the height of the coin.
   (b) Find the height of the coin after 4 seconds.
   (c) How long will it take before the coin strikes the ground?

117. **Sports** Some Major League Baseball pitchers can throw a fastball at speeds of up to and over 100 miles per hour. Assume a Major League Baseball pitcher throws a baseball straight up into the air at 100 miles per hour from a height of 6 feet 3 inches.
   (a) Use the position equation to write a mathematical model for the height of the baseball.
   (b) Find the height of the baseball after 3 seconds, 4 seconds, and 5 seconds. What must have occurred sometime in the interval $3 \leq t \leq 5$? Explain.
   (c) How many seconds is the baseball in the air?

118. **CN Tower** At 1815 feet tall, the CN Tower in Toronto, Ontario is the world's tallest self-supporting structure. An object is dropped from the top of the tower.
   (a) Use the position equation to write a mathematical model for the height of the object.
   (b) Complete the table.
   (c) From the table in part (b), determine the time interval during which the object reaches the ground. Numerically approximate the time it takes the object to reach the ground.
   (d) Find the time it takes the object to reach the ground algebraically. How close was your numerical approximation?
   (e) Use a graphing utility with the appropriate viewing window to verify your answer(s) to parts (c) and (d).
119. **Data Analysis: Movie Tickets** The average admission prices $P$ for movie theaters from 1997 to 2003 can be approximated by the model

$$P = -0.0081t^2 + 0.417t + 1.99, \quad 7 \leq t \leq 13$$

where $t$ represents the year, with $t = 7$ corresponding to 1997. (Source: Motion Picture Association of America, Inc.)

(a) Use the model to complete the table to determine when the average admission price reached or surpassed $5.

<table>
<thead>
<tr>
<th>$t$</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Verify your result from part (a) algebraically.

(c) Use the model to predict the average admission price for movie theaters in 2008. Is this prediction reasonable? How does this value compare with the admission price where you live?

120. **Data Analysis: Median Income** The median incomes $I$ (in dollars) of U.S. households from 1995 to 2002 can be approximated by the model

$$I = -133.18t^2 + 3560.5t + 19,170, \quad 5 \leq t \leq 12$$

where $t$ represents the year, with $t = 5$ corresponding to 1995. (Source: U.S. Census Bureau)

(a) Use a graphing utility to graph the model. Then use the graph to determine in which year the median income reached or surpassed $40,000.

(b) Verify your result from part (a) algebraically.

(c) Use the model to predict the median incomes of U.S. households in 2008 and 2013. Can this model be used to predict the median income of U.S. households after 2002? Why or why not?

121. **Geometry** The hypotenuse of an isosceles right triangle is 5 centimeters long. How long are its sides?

122. **Geometry** An equilateral triangle has a height of 10 inches. How long is one of its sides? (Hint: Use the height of the triangle to partition the triangle into two congruent right triangles.)

123. **Flying Speed** Two planes leave simultaneously from Chicago’s O’Hare Airport, one flying due north and the other due east (see figure). The northbound plane is flying 50 miles per hour faster than the eastbound plane. After 3 hours, the planes are 2440 miles apart. Find the speed of each plane.

124. **Boating** A winch is used to tow a boat to a dock. The rope is attached to the boat at a point 15 feet below the level of the winch (see figure).

(a) Use the Pythagorean Theorem to write an equation giving the relationship between $l$ and $x$.

(b) Find the distance from the boat to the dock when there is 75 feet of rope out.

125. **Revenue** The demand equation for a product is $p = 20 - 0.0002x$, where $p$ is the price per unit and $x$ is the number of units sold. The total revenue for selling $x$ units is

$$\text{Revenue} = xp = x(20 - 0.0002x).$$

How many units must be sold to produce a revenue of $500,000?

126. **Revenue** The demand equation for a product is $p = 60 - 0.0004x$, where $p$ is the price per unit and $x$ is the number of units sold. The total revenue for selling $x$ units is

$$\text{Revenue} = xp = x(60 - 0.0004x).$$

How many units must be sold to produce a revenue of $220,000?
Cost In Exercises 127–130, use the cost equation to find the number of units $x$ that a manufacturer can produce for the given cost $C$. Round your answer to the nearest positive integer.

127. $C = 0.125x^2 + 20x + 500 \quad C = \$14,000$
128. $C = 0.5x^2 + 15x + 5000 \quad C = \$11,500$
129. $C = 800 + 0.04x + 0.002x^2 \quad C = \$1680$
130. $C = 800 - 10x + \frac{x^2}{4} \quad C = \$896$

131. Money in Circulation The bar graph shows the total amounts of money $M$ (in billions of dollars) in circulation in the United States from 1995 to 2003. These data can be approximated by the model

$$M = 1.835t^2 + 3.58t + 333.0, \quad 5 \leq t \leq 13$$

where $t$ represents the year, with $t = 5$ corresponding to 1995. (Source: Financial Management Service, U.S. Department of the Treasury)

132. Biology The metabolic rate of an ectothermic organism increases with increasing temperature within a certain range. The graph shows experimental data for the oxygen consumption $C$ (in microliters per gram per hour) of a beetle at certain temperatures. The data can be approximated by the model

$$C = 0.45x^2 - 1.65x + 50.75, \quad 10 \leq x \leq 25$$

where $x$ is the air temperature in degrees Celsius.

(a) The oxygen consumption is 150 microliters per gram per hour. What is the air temperature?
(b) The temperature is increased from $10^\circ C$ to $20^\circ C$. The oxygen consumption is increased by approximately what factor?

133. Flying Distance A commercial jet flies to three cities whose locations form the vertices of a right triangle (see figure). The total flight distance (from Oklahoma City to Austin to New Orleans and back to Oklahoma City) is approximately 1348 miles. It is 560 miles between Oklahoma City and New Orleans. Approximate the other two distances.

(a) Use the model to complete the table to determine when the total amount of money in circulation reached or surpassed $600$ billion.

<table>
<thead>
<tr>
<th>$t$</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Verify your result from part (a) algebraically and graphically.

(c) Use the model to predict the total amount of money in circulation in 2008. Is this prediction reasonable? Explain.
134. **Geometry** An above ground swimming pool with the
dimensions shown in the figure is to be constructed such
that the volume of water in the pool is 1024 cubic feet.
The height of the pool is to be 4 feet.

(a) What are the possible dimensions of the base?
(b) One cubic foot of water weighs approximately 62.4
pounds. Find the total weight of the water in the pool.
(c) A water pump is filling the pool at a rate of 5 gallons
per minute. Find the time that will be required for the
pump to fill the pool. (Hint: One gallon of water is
approximately 0.13368 cubic foot.)

139. Solve $3(x + 4)^2 + (x + 4) - 2 = 0$ in two ways.
(a) Let $u = x + 4$, and solve the resulting equation for $u$.
   Then solve the $u$-solution for $x$.
(b) Expand and collect like terms in the equation, and
   solve the resulting equation for $x$.
(c) Which method is easier? Explain.

140. Solve each equation, given that $a$ and $b$ are not zero.
(a) $ax^2 + bx = 0$
(b) $ax^2 - ax = 0$

**Think About It** In Exercises 141–146, write a quadratic
equation that has the given solutions. (There are many
correct answers.)

| 141. $-3$ and $6$ | 142. $-4$ and $-11$ |
| 143. $8$ and $14$ | 144. $\frac{1}{2}$ and $-\frac{2}{3}$ |
| 145. $1 + \sqrt{2}$ and $1 - \sqrt{2}$ | 146. $-3 + \sqrt{5}$ and $-3 - \sqrt{5}$ |

**Skills Review**

In Exercises 147–150, identify the rule of algebra illustrated
by the statement.

| 147. $(10x)y = 10(xy)$ | 148. $-4(x - 3) = -4x + 12$ |
| 149. $7x^4 + (-7x^4) = 0$ | 150. $(x + 4) + x^3 = x + (4 + x^3)$ |

In Exercises 151–154, find the product.

| 151. $(x + 3)(x - 6)$ | 152. $(x - 8)(x - 1)$ |
| 153. $(x + 4)(x^2 - x + 2)$ | 154. $(x + 9)(x^2 - 6x + 4)$ |

In Exercises 155–158, completely factor the expression.

| 155. $x^5 - 27x^2$ | 156. $x^3 - 5x^2 - 14x$ |
| 157. $x^3 + 5x^2 - 2x - 10$ | 158. $2x^3 + x^2 - 8x - 4$ |

| 159. **Make a Decision** To work an extended application
analyzing the population of the United States, visit this
text’s website at college.hmco.com. (Data Source: U.S.
Census Bureau) |

The Make a Decision exercise indicates a multi-part exercise using large data sets. Go to college.hmco.com
to access these problems.
The Imaginary Unit $i$

In Section 1.4, you learned that some quadratic equations have no real solutions. For instance, the quadratic equation $x^2 + 1 = 0$ has no real solution because there is no real number $x$ that can be squared to produce $-1$. To overcome this deficiency, mathematicians created an expanded system of numbers using the imaginary unit $i$, defined as

$$i = \sqrt{-1}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, the set of complex numbers is obtained. Each complex number can be written in the standard form $a + bi$. For instance, the standard form of the complex number $-5 + \sqrt{-9}$ is $-5 + 3i$ because

$$-5 + \sqrt{-9} = -5 + \sqrt{3^2(-1)} = -5 + 3\sqrt{-1} = -5 + 3i.$$

In the standard form $a + bi$, the real number $a$ is called the real part of the complex number $a + bi$, and the number $bi$ (where $b$ is a real number) is called the imaginary part of the complex number.

Definition of a Complex Number

If $a$ and $b$ are real numbers, the number $a + bi$ is a complex number, and it is said to be written in standard form. If $b = 0$, the number $a + bi = a$ is a real number. If $b \neq 0$, the number $a + bi$ is called an imaginary number. A number of the form $bi$, where $b \neq 0$, is called a pure imaginary number.

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 1.22. This is true because every real number $a$ can be written as a complex number using $b = 0$. That is, for every real number $a$, you can write $a = a + 0i$.

Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other

$$a + bi = c + di$$

if and only if $a = c$ and $b = d$. 

Real numbers

Imaginary numbers

Complex numbers

FIGURE 1.22
Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

If \( a + bi \) and \( c + di \) are two complex numbers written in standard form, their sum and difference are defined as follows.

**Sum:** \( (a + bi) + (c + di) = (a + c) + (b + d)i \)

**Difference:** \( (a + bi) - (c + di) = (a - c) + (b - d)i \)

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number \( a + bi \) is

\[-(a + bi) = -a - bi.\]

So, you have

\[(a + bi) + (-a - bi) = 0 + 0i = 0.\]

**Example 1** Adding and Subtracting Complex Numbers

**a.** \((4 + 7i) + (1 - 6i) = 4 + 7i + 1 - 6i\)

\[= (4 + 1) + (7i - 6i)\]

\[= 5 + i\]

**b.** \((1 + 2i) - (4 + 2i) = 1 + 2i - 4 - 2i\)

\[= (1 - 4) + (2i - 2i)\]

\[= -3 + 0\]

\[= -3\]

**c.** \(3i - (-2 + 3i) - (2 + 5i) = 3i + 2 - 3i - 2 - 5i\)

\[= (2 - 2) + (3i - 3i - 5i)\]

\[= 0 - 5i\]

\[= -5i\]

**d.** \((3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i\)

\[= (3 + 4 - 7) + (2i - i - i)\]

\[= 0 + 0i\]

\[= 0\]

**Checkpoint** Now try Exercise 17.

Note in Examples 1(b) and 1(d) that the sum of two complex numbers can be a real number.
Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

**Associative Properties of Addition and Multiplication**

**Commutative Properties of Addition and Multiplication**

**Distributive Property of Multiplication Over Addition**

Notice below how these properties are used when two complex numbers are multiplied.

\[
(a + bi)(c + di) = ac + (ad)i + (bc)i + (bd)i^2
\]

Distributive Property

\[
= ac + (ad)i + (bc)i + (bd)(-1)
\]

Distributive Property

\[
i^2 = -1
\]

\[
= ac - bd + (ad)i + (bc)i
\]

Commutative Property

\[
= (ac - bd) + (ad + bc)i
\]

Associative Property

Rather than trying to memorize this multiplication rule, you should simply remember how the Distributive Property is used to multiply two complex numbers.

**Example 2**  

**Multiplying Complex Numbers**

- **a.**  
  \[
  4(-2 + 3i) = 4(-2) + 4(3i)
  = -8 + 12i
  \]
  
  Distributive Property
  
  **Simplify.**

- **b.**  
  \[
  (2 - i)(4 + 3i) = 2(4 + 3i) - i(4 + 3i)
  = 8 + 6i - 4i - 3i^2
  = 8 + 6i - 4i - 3(-1)
  = (8 + 3) + (6i - 4i)
  = 11 + 2i
  \]
  
  Distributive Property
  
  **Distributive Property**
  
  \[
i^2 = -1
\]

Group like terms.

Write in standard form.

- **c.**  
  \[
  (3 + 2i)(3 - 2i) = 3(3 - 2i) + 2i(3 - 2i)
  = 9 - 6i + 6i - 4i^2
  = 9 - 6i + 6i - 4(-1)
  = 9 + 4
  = 13
  \]
  
  Distributive Property
  
  **Distributive Property**
  
  \[
i^2 = -1
\]

Simplify.

Write in standard form.

- **d.**  
  \[
  (3 + 2i)^2 = (3 + 2i)(3 + 2i)
  = 3(3 + 2i) + 2i(3 + 2i)
  = 9 + 6i + 6i + 4i^2
  = 9 + 6i + 6i + 4(-1)
  = 9 + 12i - 4
  = 5 + 12i
  \]
  
  Square of a binomial
  
  Distributive Property
  
  **Distributive Property**
  
  \[
i^2 = -1
\]

Simplify.

Write in standard form.

**CHECKPOINT**

Now try Exercise 27.
Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the form \(a + bi\) and \(a - bi\), called complex conjugates.

\[(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 - b^2(-1) = a^2 + b^2\]

**Example 3** Multiplying Conjugates

Multiply each complex number by its complex conjugate.

**a.** \(1 + i\)  \n**b.** \(4 - 3i\)

**Solution**

**a.** The complex conjugate of \(1 + i\) is \(1 - i\).

\[(1 + i)(1 - i) = 1^2 - i^2 = 1 - (-1) = 2\]

**b.** The complex conjugate of \(4 - 3i\) is \(4 + 3i\).

\[(4 - 3i)(4 + 3i) = 4^2 - (3i)^2 = 16 - 9i^2 = 16 - 9(-1) = 25\]

Now try Exercise 37.

**STUDY TIP**

Note that when you multiply the numerator and denominator of a quotient of complex numbers by \(\frac{c - di}{c - di}\), you are actually multiplying the quotient by a form of 1. You are not changing the original expression, you are only creating an expression that is equivalent to the original expression.

To write the quotient of \(a + bi\) and \(c + di\) in standard form, where \(c\) and \(d\) are not both zero, multiply the numerator and denominator by the complex conjugate of the denominator to obtain

\[\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.\]

**Example 4** Writing a Quotient of Complex Numbers in Standard Form

\[\frac{2 + 3i}{4 - 2i} = \frac{2 + 3i(4 + 2i)}{4 - 2i(4 + 2i)}\]

Multiply numerator and denominator by complex conjugate of denominator.

\[= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2}\]

Expand.

\[= \frac{8 - 6 + 16i}{16 + 4}\]

\[i^2 = -1\]

Simplify.

\[= \frac{2 + 16i}{20}\]

Write in standard form.

\[= \frac{1}{10} + \frac{4}{5}i\]

Now try Exercise 49.
Complex Solutions of Quadratic Equations

When using the Quadratic Formula to solve a quadratic equation, you often obtain a result such as \( \sqrt{-3} \), which you know is not a real number. By factoring out \( i = \sqrt{-1} \), you can write this number in standard form:

\[
\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}i
\]

The number \( \sqrt{3}i \) is called the principal square root of \( -3 \).

Principal Square Root of a Negative Number

If \( a \) is a positive number, the principal square root of the negative number \( -a \) is defined as

\[
\sqrt{-a} = \sqrt{a}i.
\]

Example 5 Writing Complex Numbers in Standard Form

a. \( \sqrt{-3}\sqrt{-12} = \sqrt{3}i\sqrt{12}i = \sqrt{36}i^2 = 6(-1) = -6 \)
b. \( \sqrt{-48} - \sqrt{-27} = 4\sqrt{3}i - 3\sqrt{3}i = \sqrt{3}i \)
c. \( (-1 + \sqrt{-3})^2 = (-1 + \sqrt{3}i)^2 \\
= (-1)^2 - 2\sqrt{3}i + (\sqrt{3})^2(i^2) \\
= 1 - 2\sqrt{3}i + 3(-1) \\
= -2 - 2\sqrt{3}i \)

CHECKPOINT Now try Exercise 59.

Example 6 Complex Solutions of a Quadratic Equation

Solve (a) \( x^2 + 4 = 0 \) and (b) \( 3x^2 - 2x + 5 = 0 \).

Solution

a. \( x^2 + 4 = 0 \)

\[
x^2 = -4
\]

\[
x = \pm 2i
\]

Write original equation.

Subtract 4 from each side.

Extract square roots.

b. \( 3x^2 - 2x + 5 = 0 \)

\[
x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}
\]

\[
\frac{2 \pm \sqrt{-56}}{6}
\]

Quadratic Formula

Simplify.

Write \( \sqrt{-56} \) in standard form.

Write in standard form.

CHECKPOINT Now try Exercise 65.
1.5  Exercises

VOCABULARY CHECK:

1. Match the type of complex number with its definition.
   (a) Real Number
   (b) Imaginary number
   (c) Pure imaginary number

   (i) \( a + bi \), \( a \neq 0 \), \( b \neq 0 \)
   (ii) \( a + bi \), \( a = 0 \), \( b \neq 0 \)
   (iii) \( a + bi \), \( b = 0 \)

   In Exercises 2–4, fill in the blanks.
   2. The imaginary unit \( i \) is defined as \( i = \ldots \), where \( i^2 = \ldots \).
   3. If \( a \) is a positive number, the \( \ldots \) root of the negative number \( -a \) is defined as \( \sqrt{-a} = \sqrt{a}i \).
   4. The numbers \( a + bi \) and \( a - bi \) are called \( \ldots \) \( \ldots \), and their product is a real number \( a^2 + b^2 \).


In Exercises 1–4, find real numbers \( a \) and \( b \) such that the equation is true.
1. \( a + bi = -10 + 6i \)
2. \( a + bi = 13 + 4i \)
3. \( (a - 1) + (b + 3)i = 5 + 8i \)
4. \( (a + 6) + 2bi = 6 - 5i \)

In Exercises 5–16, write the complex number in standard form.
5. \( 4 + \sqrt{-9} \)
6. \( 3 + \sqrt{-16} \)
7. \( 2 - \sqrt{-27} \)
8. \( 1 + \sqrt{-8} \)
9. \( \sqrt{-75} \)
10. \( \sqrt{-4} \)
11. \( 8 \)
12. \( 45 \)
13. \( -6i + i^2 \)
14. \( -4i^2 + 2i \)
15. \( \sqrt{-0.09} \)
16. \( \sqrt{-0.0004} \)

In Exercises 17–26, perform the addition or subtraction and write the result in standard form.
17. \( (5 + i) + (6 - 2i) \)
18. \( (13 - 2i) + (-5 + 6i) \)
19. \( (8 - i) - (4 - i) \)
20. \( (3 + 2i) - (6 + 13i) \)
21. \( -2 + \sqrt{-8} + (5 - \sqrt{-50}) \)
22. \( 8 + \sqrt{-18} - (4 + 3\sqrt{2}i) \)
23. \( 13i - (14 - 7i) \)
24. \( 22 + (-5 + 8i) + 10i \)
25. \( -\left(\frac{3}{2} + \frac{5}{3}\right)i + \left(\frac{1}{2} + \frac{1}{4}\right)i \)
26. \( (1.6 + 3.2i) + (-5.8 + 4.3i) \)

In Exercises 27–36, perform the operation and write the result in standard form.
27. \( (1 + i)(3 - 2i) \)
28. \( (6 - 2i)(2 - 3i) \)
29. \( 6i(5 - 2i) \)
30. \( -8i(9 + 4i) \)
31. \( (\sqrt{14} + \sqrt{10}i)(\sqrt{14} - \sqrt{10}i) \)
32. \( (\sqrt{3} + \sqrt{15}i)(\sqrt{3} - \sqrt{15}i) \)
33. \( (4 + 5i)^2 \)
34. \( (2 - 3i)^2 \)
35. \( (2 + 3i)^2 + (2 - 3i)^2 \)
36. \( (1 - 2i)^2 - (1 + 2i)^2 \)

In Exercises 37–44, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.
37. \( 6 + 3i \)
38. \( 7 - 12i \)
39. \( -1 - \sqrt{3}i \)
40. \( -3 + \sqrt{2}i \)
41. \( \sqrt{-20} \)
42. \( \sqrt{-15} \)
43. \( \sqrt{8} \)
44. \( 1 + \sqrt{8} \)

In Exercises 45–54, write the quotient in standard form.
45. \( \frac{5}{i} \)
46. \( \frac{14}{2i} \)
47. \( \frac{2}{4 - 5i} \)
48. \( \frac{5}{1 - i} \)
49. \( \frac{3 + i}{3 - i} \)
50. \( \frac{6 - 7i}{1 - 2i} \)
51. \( \frac{6 - 5i}{i} \)
52. \( \frac{8 + 16i}{2i} \)
53. \( \frac{3i}{(4 - 5i)^2} \)
54. \( \frac{5i}{(2 + 3i)^2} \)

In Exercises 55–58, perform the operation and write the result in standard form.
55. \( \frac{2}{1 + i} - \frac{3}{1 - i} \)
56. \( \frac{2i}{2 + i} + \frac{5}{2 - i} \)
57. \( \frac{i}{3 - 2i} + \frac{2i}{3 + 8i} \)
58. \( \frac{1 + i}{i} - \frac{3}{4 - i} \)
In Exercises 59–64, write the complex number in standard form.

59. \( \sqrt{-6} \cdot \sqrt{-2} \)  
60. \( \sqrt{-5} \cdot \sqrt{-10} \)

61. \( (\sqrt{-10})^2 \)  
62. \( (\sqrt{-75})^2 \)

63. \( (3 + \sqrt{-5})(7 - \sqrt{-10}) \)  
64. \( (2 - \sqrt{-6})^2 \)

In Exercises 65–74, use the Quadratic Formula to solve the quadratic equation.

65. \( x^2 - 2x + 2 = 0 \)  
66. \( x^2 + 6x + 10 = 0 \)

67. \( 4x^2 + 16x + 17 = 0 \)  
68. \( 9x^2 - 6x + 37 = 0 \)

69. \( 4x^2 + 16x + 15 = 0 \)  
70. \( 16x^2 - 4t + 3 = 0 \)

71. \( \frac{3}{2}x^2 - 6x + 9 = 0 \)  
72. \( \frac{7}{8}x^2 - \frac{3}{4}x + \frac{5}{16} = 0 \)

73. \( 1.4x^2 - 2x - 10 = 0 \)  
74. \( 4.5x^2 - 3x + 12 = 0 \)

In Exercises 75–82, simplify the complex number and write it in standard form.

75. \( -6i^3 + i^2 \)  
76. \( 4i^2 - 2i^3 \)

77. \( -5i^5 \)  
78. \( -(i)^3 \)

79. \( (\sqrt{-75})^3 \)  
80. \( (\sqrt{-2})^6 \)

81. \( \frac{1}{i^3} \)  
82. \( \frac{1}{(2i)^3} \)

83. **Impedance**  The opposition to current in an electrical circuit is called its impedance. The impedance \( z \) in a parallel circuit with two pathways satisfies the equation

\[
\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}
\]

where \( z_1 \) is the impedance (in ohms) of pathway 1 and \( z_2 \) is the impedance of pathway 2.

(a) The impedance of each pathway in a parallel circuit is found by adding the impedances of all components in the pathway. Use the table to find \( z_1 \) and \( z_2 \).

(b) Find the impedance \( z \).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Resistor</th>
<th>Inductor</th>
<th>Capacitor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Impedance</td>
<td>( a \omega )</td>
<td>( bi \omega )</td>
<td>( -ci \omega )</td>
</tr>
</tbody>
</table>

84. Cube each complex number.
   (a) \( 2 \)  
   (b) \( -1 + \sqrt{3}i \)  
   (c) \( -1 - \sqrt{3}i \)

85. Raise each complex number to the fourth power.
   (a) \( 2 \)  
   (b) \( -2 \)  
   (c) \( 2i \)  
   (d) \( -2i \)

86. Write each of the powers of \( i \) as \( i, -i, 1, \) or \( -1 \).
   (a) \( i^{40} \)  
   (b) \( i^{25} \)  
   (c) \( i^{50} \)  
   (d) \( i^{67} \)

**Synthesis**

**True or False?** In Exercises 87–89, determine whether the statement is true or false. Justify your answer.

87. There is no complex number that is equal to its complex conjugate.
88. \( -i\sqrt{6} \) is a solution of \( x^4 - x^2 + 14 = 56 \).
89. \( i^{44} + i^{50} - i^{74} - i^{109} + i^{64} = -1 \)

90. **Error Analysis**  Describe the error.

\[
\sqrt{-6} - \sqrt{-6} = \sqrt{(-6)(-6)} = \sqrt{36} = 6
\]

91. **Proof**  Prove that the complex conjugate of the product of two complex numbers \( a_1 + b_1i \) and \( a_2 + b_2i \) is the product of their complex conjugates.
92. **Proof**  Prove that the complex conjugate of the sum of two complex numbers \( a_1 + b_1i \) and \( a_2 + b_2i \) is the sum of their complex conjugates.

**Skills Review**

In Exercises 93–96, perform the operation and write the result in standard form.

93. \( (4 + 3x) + (8 - 6x - x^2) \)  
94. \( (x^3 - 3x^2) - (6 - 2x - 4x^2) \)

95. \( (3x - \frac{1}{2})(x + 4) \)  
96. \( (2x - 5)^2 \)

In Exercises 97–100, solve the equation and check your solution.

97. \( -x - 12 = 19 \)  
98. \( 8 - 3x = -34 \)
99. \( 4(5x - 6) - 3(6x + 1) = 0 \)
100. \( 5[x - (3x + 11)] = 20x - 15 \)

101. **Volume of an Oblate Spheroid**  Solve for \( a : V = \frac{4}{3}pi^2b \)

102. **Newton’s Law of Universal Gravitation**

Solve for \( r : F = \frac{m_1m_2}{r^2} \)

103. **Mixture Problem**  A five-liter container contains a mixture with a concentration of 50%. How much of this mixture must be withdrawn and replaced by 100% concentrate to bring the mixture up to 60% concentration?
Section 1.6 Other Types of Equations

What you should learn
• Solve polynomial equations of degree three or greater.
• Solve equations involving radicals.
• Solve equations involving fractions or absolute values.
• Use polynomial equations and equations involving radicals to model and solve real-life problems.

Why you should learn it
Polynomial equations, radical equations, and absolute value equations can be used to model and solve real-life problems. For instance, in Exercise 96 on page 142, a radical equation can be used to model the total monthly cost of airplane flights between Chicago and Denver.

Polynomial Equations

In this section you will extend the techniques for solving equations to nonlinear and nonquadratic equations. At this point in the text, you have only four basic methods for solving nonlinear equations—factoring, extracting square roots, completing the square, and the Quadratic Formula. So the main goal of this section is to learn to rewrite nonlinear equations in a form to which you can apply one of these methods.

Example 1 shows how to use factoring to solve a polynomial equation, which is an equation that can be written in the general form

\[ a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0. \]

Example 1

Solving a Polynomial Equation by Factoring

Solve \(3x^4 = 48x^2\).

Solution

First write the polynomial equation in general form with zero on one side, factor the other side, and then set each factor equal to zero and solve.

\[
3x^4 = 48x^2 \\
3x^4 - 48x^2 = 0 \\
3x^2(x^2 - 16) = 0 \\
3x^2(x + 4)(x - 4) = 0
\]

Write original equation.
Write in general form.
Factor out common factor.
Write in factored form.

\[
x = 0 \\
x + 4 = 0 \\
x - 4 = 0
\]

Set 1st factor equal to 0.
Set 2nd factor equal to 0.
Set 3rd factor equal to 0.

You can check these solutions by substituting in the original equation, as follows.

Check

\[
3(0)^4 = 48(0)^2 \quad 0 \text{ checks. } \checkmark \\
3(-4)^4 = 48(-4)^2 \quad -4 \text{ checks. } \checkmark \\
3(4)^4 = 48(4)^2 \quad 4 \text{ checks. } \checkmark
\]

So, you can conclude that the solutions are \(x = 0\), \(x = -4\), and \(x = 4\).

CHECKPOINT Now try Exercise 1.

A common mistake that is made in solving an equation like that in Example 1 is to divide each side of the equation by the variable factor \(x^2\). This loses the solution \(x = 0\). When solving an equation, always write the equation in general form, then factor the equation and set each factor equal to zero. Do not divide each side of an equation by a variable factor in an attempt to simplify the equation.
For a review of factoring special polynomial forms, see Section P.4.

**Example 2  Solving a Polynomial Equation by Factoring**

Solve $x^3 - 3x^2 + 3x - 9 = 0$.

**Solution**

Write original equation.

$x^3 - 3x^2 + 3x - 9 = 0$

Factor by grouping.

$x^2(x - 3) + 3(x - 3) = 0$

Distributive Property

$(x - 3)(x^2 + 3) = 0$

Set 1st factor equal to 0.

$x - 3 = 0 \quad \Rightarrow \quad x = 3$

Set 2nd factor equal to 0.

$x^2 + 3 = 0 \quad \Rightarrow \quad x = \pm \sqrt{3}i$

The solutions are $x = 3$, $x = \sqrt{3}i$, and $x = -\sqrt{3}i$.

**CHECKPOINT**  Now try Exercise 9.

Occasionally, mathematical models involve equations that are of **quadratic type**. In general, an equation is of quadratic type if it can be written in the form

$$au^2 + bu + c = 0$$

where $a \neq 0$ and $u$ is an algebraic expression.

**Example 3  Solving an Equation of Quadratic Type**

Solve $x^4 - 3x^2 + 2 = 0$.

**Solution**

This equation is of quadratic type with $u = x^2$.

$$(x^2)^2 - 3(x^2) + 2 = 0$$

To solve this equation, you can factor the left side of the equation as the product of two second-degree polynomials.

$$x^4 - 3x^2 + 2 = 0$$

Write original equation.

$$u^2 - 3u + 2 = 0$$

Quadratic form

$$(x^2)^2 - 3(x^2) + 2 = 0$$

Partially factor.

$$(x^2 - 1)(x^2 - 2) = 0$$

Factor completely.

$x + 1 = 0 \quad \Rightarrow \quad x = -1$

Set 1st factor equal to 0.

$x - 1 = 0 \quad \Rightarrow \quad x = 1$

Set 2nd factor equal to 0.

$x^2 - 2 = 0 \quad \Rightarrow \quad x = \pm \sqrt{2}$

Set 3rd factor equal to 0.

The solutions are $x = -1$, $x = 1$, $x = \sqrt{2}$, and $x = -\sqrt{2}$. Check these in the original equation.

**CHECKPOINT**  Now try Exercise 13.
Equations Involving Radicals

Operations such as squaring each side of an equation, raising each side of an equation to a rational power, and multiplying each side of an equation by a variable quantity all can introduce extraneous solutions. So, when you use any of these operations, checking your solutions is crucial.

Example 4  Solving Equations Involving Radicals

a. \( \sqrt{2x + 7} - x = 2 \)

\[
\sqrt{2x + 7} = x + 2
\]

\[
2x + 7 = x^2 + 4x + 4
\]

0 = \( x^2 + 2x - 3 \)

0 = \( (x + 3)(x - 1) \)

\[
x + 3 = 0 \quad \Rightarrow \quad x = -3
\]

\[
x - 1 = 0 \quad \Rightarrow \quad x = 1
\]

By checking these values, you can determine that the only solution is \( x = 1 \).

b. \( \sqrt{2x - 5} - \sqrt{x - 3} = 1 \)

\[
\sqrt{2x - 5} = \sqrt{x - 3} + 1
\]

\[
2x - 5 = x - 3 + 2\sqrt{x - 3} + 1
\]

\[
2x - 5 = x - 2 + 2\sqrt{x - 3}
\]

\[
x - 3 = 2\sqrt{x - 3}
\]

\[
x^2 - 6x + 9 = 4(x - 3)
\]

\[
x^2 - 10x + 21 = 0
\]

\[
(x - 3)(x - 7) = 0
\]

\[
x - 3 = 0 \quad \Rightarrow \quad x = 3
\]

\[
x - 7 = 0 \quad \Rightarrow \quad x = 7
\]

The solutions are \( x = 3 \) and \( x = 7 \). Check these in the original equation.

Example 5  Solving an Equation Involving a Rational Exponent

\[
(x - 4)^{2/3} = 25
\]

\[
\sqrt[3]{(x - 4)^2} = 25
\]

\[
(x - 4)^2 = 15,625
\]

\[
x - 4 = \pm 125
\]

\[
x = 129, \quad x = -121
\]

Now try Exercise 31.

When an equation contains two radicals, it may not be possible to isolate both. In such cases, you may have to raise each side of the equation to a power at two different stages in the solution, as shown in Example 4(b).
Equations with Fractions or Absolute Values

To solve an equation involving fractions, multiply each side of the equation by the least common denominator (LCD) of all terms in the equation. This procedure will “clear the equation of fractions.” For instance, in the equation
\[
\frac{2}{x^2 + 1} + \frac{1}{x} = \frac{2}{x}
\]
you can multiply each side of the equation by \(x(x^2 + 1)\). Try doing this and solve the resulting equation. You should obtain one solution: \(x = 1\).

**Example 6  Solving an Equation Involving Fractions**

Solve \(\frac{2}{x} = \frac{3}{x - 2} - 1\).

**Solution**

For this equation, the least common denominator of the three terms is \(x(x - 2)\), so you begin by multiplying each term of the equation by this expression.

\[
\frac{2}{x} = \frac{3}{x - 2} - 1
\]

Write original equation.

\[
x(x - 2) \cdot \frac{2}{x} = x(x - 2) \cdot \frac{3}{x - 2} - x(x - 2) \cdot 1
\]

Multiply each term by the LCD.

\[
2(x - 2) = 3x - x(x - 2)
\]

Simplify.

\[
2x - 4 = -x^2 + 5x
\]

Simplify.

\[
x^2 - 3x - 4 = 0
\]

Write in general form.

\[
(x - 4)(x + 1) = 0
\]

Factor.

\[
x - 4 = 0 \quad \Rightarrow \quad x = 4
\]

Set 1st factor equal to 0.

\[
x + 1 = 0 \quad \Rightarrow \quad x = -1
\]

Set 2nd factor equal to 0.

**Check** \(x = 4\)

\[
\frac{2}{x} = \frac{3}{x - 2} - 1
\]

\[
\frac{2}{4} = \frac{3}{4 - 2} - 1
\]

\[
\frac{1}{2} = \frac{3}{2} - 1
\]

\[
\frac{1}{2} = 1
\]

\(
\checkmark
\)

**Check** \(x = -1\)

\[
\frac{2}{x} = \frac{3}{x - 2} - 1
\]

\[
\frac{2}{-1} = \frac{3}{-1 - 2} - 1
\]

\[
-2 = -1 - 1
\]

\[
-2 = -2 \checkmark
\]

So, the solutions are \(x = 4\) and \(x = -1\).

**CHECKPOINT** Now try Exercise 59.
To solve an equation involving an absolute value, remember that the expression inside the absolute value signs can be positive or negative. This results in two separate equations, each of which must be solved. For instance, the equation

$$|x - 2| = 3$$

results in the two equations $x - 2 = 3$ and $-(x - 2) = 3$, which implies that the equation has two solutions: $x = 5$ and $x = -1$.

### Example 7 Solving an Equation Involving Absolute Value

Solve $|x^2 - 3x| = -4x + 6$.

**Solution**

Because the variable expression inside the absolute value signs can be positive or negative, you must solve the following two equations.

**First Equation**

$$x^2 - 3x = -4x + 6$$

Use positive expression.

$$x^2 + x - 6 = 0$$

Write in general form.

$$(x + 3)(x - 2) = 0$$

Factor.

$$x + 3 = 0$$  $\Rightarrow$  $x = -3$

Set 1st factor equal to 0.

$$x - 2 = 0$$  $\Rightarrow$  $x = 2$

Set 2nd factor equal to 0.

**Second Equation**

$$-(x^2 - 3x) = -4x + 6$$

Use negative expression.

$$x^2 - 7x + 6 = 0$$

Write in general form.

$$(x - 1)(x - 6) = 0$$

Factor.

$$x - 1 = 0$$  $\Rightarrow$  $x = 1$

Set 1st factor equal to 0.

$$x - 6 = 0$$  $\Rightarrow$  $x = 6$

Set 2nd factor equal to 0.

**Check**

$$|(-3)^2 - 3(-3)| = ? -4(-3) + 6$$

$$18 = 18$$

$-3$ checks. $\checkmark$

$$|(2)^2 - 3(2)| = ? -4(2) + 6$$

$$2 \neq -2$$

$2$ does not check.

$$|(1)^2 - 3(1)| = ? -4(1) + 6$$

$$2 = 2$$

$1$ checks. $\checkmark$

$$|(6)^2 - 3(6)| = ? -4(6) + 6$$

$$18 \neq -18$$

$6$ does not check.

The solutions are $x = -3$ and $x = 1$.

Now try Exercise 67.
Applications

It would be impossible to categorize the many different types of applications that involve nonlinear and nonquadratic models. However, from the few examples and exercises that are given, you will gain some appreciation for the variety of applications that can occur.

Example 8 Reduced Rates

A ski club chartered a bus for a ski trip at a cost of $480. In an attempt to lower the bus fare per skier, the club invited nonmembers to go along. After five nonmembers joined the trip, the fare per skier decreased by $4.80. How many club members are going on the trip?

Solution

Begin the solution by creating a verbal model and assigning labels.

Verbal Model: Cost per skier \( \cdot \) Number of skiers = Cost of trip

Labels:
- Cost of trip = 480 (dollars)
- Number of ski club members = \( x \) (people)
- Number of skiers = \( x + 5 \) (people)
- Original cost per member = \( \frac{480}{x} \) (dollars per person)
- Cost per skier = \( \frac{480}{x} - 4.80 \) (dollars per person)

Equation:

\[
\left( \frac{480}{x} - 4.80 \right)(x + 5) = 480
\]

\[
\left( \frac{480 - 4.8x}{x} \right)(x + 5) = 480
\]

\[
480 - 4.8x(x + 5) = 480x
\]

\[
480x + 2400 - 4.8x^2 - 24x = 480x
\]

\[
-4.8x^2 - 24x + 2400 = 0
\]

\[
x^2 + 5x - 500 = 0
\]

\[
(x + 25)(x - 20) = 0
\]

\[
x + 25 = 0 \quad \Rightarrow \quad x = -25
\]

\[
x - 20 = 0 \quad \Rightarrow \quad x = 20
\]

Choosing the positive value of \( x \), you can conclude that 20 ski club members are going on the trip. Check this in the original statement of the problem, as follows.

\[
\left( \frac{480}{20} - 4.80 \right)(20 + 5) = 480
\]

\[
(24 - 4.80)25 = 480
\]

480 = 480

20 checks. ✓

Now try Exercise 87.
Interest in a savings account is calculated by one of three basic methods: simple interest, interest compounded \( n \) times per year, and interest compounded continuously. The next example uses the formula for interest that is compounded \( n \) times per year.

\[
A = P \left(1 + \frac{r}{n}\right)^{nt}
\]

In this formula, \( A \) is the balance in the account, \( P \) is the principal (or original deposit), \( r \) is the annual interest rate (in decimal form), \( n \) is the number of compoundings per year, and \( t \) is the time in years. In Chapter 5, you will study a derivation of the formula above for interest compounded continuously.

**Example 9  Compound Interest**

When you were born, your grandparents deposited $5000 in a long-term investment in which the interest was compounded quarterly. Today, on your 25th birthday, the value of the investment is $25,062.59. What is the annual interest rate for this investment?

**Solution**

**Formula:**

\[
A = P \left(1 + \frac{r}{n}\right)^{nt}
\]

**Labels:**

- Balance = \( A = 25,062.59 \) (dollars)
- Principal = \( P = 5000 \) (dollars)
- Time = \( t = 25 \) (years)
- Compoundings per year = \( n = 4 \) (compoundings per year)
- Annual interest rate = \( r \) (percent in decimal form)

**Equation:**

\[
25,062.59 = 5000 \left(1 + \frac{r}{4}\right)^{4(25)}
\]

\[
\frac{25,062.59}{5000} = \left(1 + \frac{r}{4}\right)^{100}
\]

Divide each side by 5000.

\[
5.0125 \approx \left(1 + \frac{r}{4}\right)^{100}
\]

Use a calculator.

\[
(5.0125)^{1/100} = 1 + \frac{r}{4}
\]

Raise each side to reciprocal power.

\[
1.01625 \approx 1 + \frac{r}{4}
\]

Use a calculator.

\[
0.01625 = \frac{r}{4}
\]

Subtract 1 from each side.

\[
0.065 = r
\]

Multiply each side by 4.

The annual interest rate is about 0.065, or 6.5%. Check this in the original statement of the problem.

**CHECKPOINT**

Now try Exercise 91.
1.6 Exercises

**VOCABULARY CHECK:** Fill in the blanks.
1. The equation $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0$ is a _______ equation in $x$ written in general form.
2. Squaring each side of an equation, multiplying each side of an equation by a variable quantity, and raising each side of an equation to a rational power are all operations that can introduce _______ solutions to a given equation.
3. The equation $2x^4 + x^2 + 1 = 0$ is of _______ _______.

**PREREQUISITE SKILLS REVIEW:** Practice and review algebra skills needed for this section at [www.Eduspace.com](http://www.Eduspace.com).

In Exercises 1–24, find all solutions of the equation. Check your solutions in the original equation.

1. $4x^4 - 18x^2 = 0$
2. $20x^3 - 125x = 0$
3. $x^4 - 81 = 0$
4. $x^6 - 64 = 0$
5. $x^3 + 216 = 0$
6. $27x^3 - 512 = 0$
7. $5x^3 + 30x^2 + 45x = 0$
8. $9x^4 - 24x^3 + 16x^2 = 0$
9. $x^3 - 3x^2 - x + 3 = 0$
10. $x^3 + 2x^2 + 3x + 6 = 0$
11. $x^4 - x^3 + x - 1 = 0$
12. $x^4 + 2x^3 - 8x - 16 = 0$
13. $x^4 - 4x^2 + 3 = 0$
14. $x^6 + 5x^2 - 36 = 0$
15. $4x^4 - 65x^2 + 16 = 0$
16. $36x^4 + 29x^2 - 7 = 0$
17. $x^6 + 7x^3 - 8 = 0$
18. $x^6 + 3x^2 + 2 = 0$
19. \( \frac{1}{x^2} + \frac{8}{x} + 15 = 0 \)
20. $6\left(\frac{x}{x+1}\right)^2 + 5\left(\frac{x}{x+1}\right) - 6 = 0$
21. $2x + 9\sqrt{x} = 5$
22. $6x - 7\sqrt{x} - 3 = 0$
23. $3x^{4/3} + 2x^{2/3} = 5$
24. $9x^{2/3} + 24x^{1/3} + 16 = 0$

**Graphical Analysis** In Exercises 25–28, (a) use a graphing utility to graph the equation, (b) use the graph to approximate any $x$-intercepts of the graph, (c) set $y = 0$ and solve the resulting equation, and (d) compare the result of part (c) with the $x$-intercepts of the graph.

25. $y = x^3 - 2x^2 - 3x$
26. $y = 2x^4 - 15x^3 + 18x^2$
27. $y = x^4 - 10x^2 + 9$
28. $y = x^4 - 29x^2 + 100$

In Exercises 29–52, find all solutions of the equation. Check your solutions in the original equation.

29. $\sqrt{2x} - 10 = 0$
30. $4\sqrt{x} - 3 = 0$
31. $\sqrt{x - 10} - 4 = 0$
32. $\sqrt{5} \cdot \sqrt{x} - 3 = 0$
33. $\frac{1}{2}x + 3 + 3 = 0$
34. $\sqrt{3x + 1} - 5 = 0$
35. $-\sqrt{26} - 11x + 4 = x$
36. $x + \sqrt{31} - 9x = 5$
37. $\sqrt{x + 1} = \sqrt{3x + 1}$
38. $\sqrt{x + 5} = \sqrt{x - 5}$
39. $\sqrt{x} - \sqrt{x - 5} = 1$
40. $\sqrt{x} + \sqrt{x - 20} = 10$
41. $\sqrt{x + 5} + \sqrt{x - 5} = 10$
42. $2\sqrt{x + 1} - \sqrt{2x + 3} = 1$
43. $\sqrt{x + 2} - \sqrt{2x - 3} = -1$
44. $4\sqrt{x - 3} - \sqrt{6x - 17} = 3$
45. $(x - 5)^{3/2} = 8$
46. $(x + 3)^{3/2} = 8$
47. $(x + 3)^{2/3} = 8$
48. $(x + 2)^{2/3} = 9$
49. $(x^2 - 5)^{3/2} = 27$
50. $(x^2 - x - 22)^{3/2} = 27$
51. $3(x - 1)^{1/2} + 2(x - 1)^{3/2} = 0$
52. $4x^2(x - 1)^{1/3} + 6x(x - 1)^{3/2} = 0$

**Graphical Analysis** In Exercises 53–56, (a) use a graphing utility to graph the equation, (b) use the graph to approximate any $x$-intercepts of the graph, (c) set $y = 0$ and solve the resulting equation, and (d) compare the result of part (c) with the $x$-intercepts of the graph.

53. $y = \sqrt{11x - 30} - x$
54. $y = 2x - \sqrt{15 - 4x}$
55. $y = \sqrt{7x + 36} - \sqrt{5x + 16} - 2$
56. $y = 3\sqrt{x} - \frac{4}{\sqrt{x}} - 4$
In Exercises 57–70, find all solutions of the equation. Check your solutions in the original equation.

57. \( x = \frac{3}{x} + \frac{1}{2} \)

58. \( \frac{4}{x} - \frac{5}{3} = \frac{x}{6} \)

59. \( \frac{1}{x} - \frac{1}{x + 1} = 3 \)

60. \( \frac{4}{x + 1} - \frac{3}{x + 2} = 1 \)

61. \( \frac{20 - x}{x} = x \)

62. \( 4x + 1 = \frac{3}{x} \)

63. \( \frac{x}{x^2 - 4} + \frac{1}{x + 2} = 3 \)

64. \( \frac{x + 1}{3} - \frac{x + 1}{x + 2} = 0 \)

65. \( |2x - 1| = 5 \)

66. \( |3x + 2| = 7 \)

67. \( |x| = x^2 + x - 3 \)

68. \( |x^2 + 6x| = 3x + 18 \)

69. \( |x + 1| = x^2 - 5 \)

70. \( |x - 10| = x^2 - 10x \)

**Graphical Analysis**  In Exercises 71–74, (a) use a graphing utility to graph the equation, (b) use the graph to approximate any x-intercepts of the graph, (c) set \( y = 0 \) and solve the resulting equation, and (d) compare the result of part (c) with the x-intercepts of the graph.

71. \( y = \frac{1}{x} - \frac{4}{x - 1} - 1 \)

72. \( y = x + \frac{9}{x + 1} - 5 \)

73. \( y = |x + 1| - 2 \)

74. \( y = |x - 2| - 3 \)

In Exercises 75–78, find the real solutions of the equation algebraically. (Round your answers to three decimal places.)

75. \( 3.2x^4 - 1.5x^2 - 2.1 = 0 \)

76. \( 7.08x^6 + 4.15x^3 - 9.6 = 0 \)

77. \( 1.8x - 6\sqrt{x} - 5.6 = 0 \)

78. \( 4x^{2/3} + 8x^{1/3} + 3.6 = 0 \)

**Think About It**  In Exercises 79–86, find an equation that has the given solutions. (There are many correct answers.)

79. \(-2, 5\)  
80. \(0, 3, 5\)

81. \(-\frac{7}{3}, 5\)  
82. \(-\frac{1}{5}, -\frac{4}{5}\)

83. \(\sqrt{3}, -\sqrt{3}, 4\)

84. \(2\sqrt{7}, -\sqrt{7}\)

85. \(-1, 1, i, -i\)

86. \(4i, -4i, 6, -6\)

87. **Chartering a Bus**  A college charters a bus for $1700 to take a group to a museum. When six more students join the trip, the cost per student drops by $7.50. How many students were in the original group?

88. **Renting an Apartment**  Three students are planning to rent an apartment for a year and share equally in the cost. By adding a fourth person, each person could save $75 a month. How much is the monthly rent?

89. **Airspeed**  An airline runs a commuter flight between Portland, Oregon and Seattle, Washington, which are 145 miles apart. If the average speed of the plane could be increased by 40 miles per hour, the travel time would be decreased by 12 minutes. What airspeed is required to obtain this decrease in travel time?

90. **Average Speed**  A family drove 1080 miles to their vacation lodge. Because of increased traffic density, their average speed on the return trip was decreased by 6 miles per hour and the trip took 2\frac{1}{2} hours longer. Determine their average speed on the way to the lodge.

91. **Mutual Funds**  A deposit of $2500 in a mutual fund reaches a balance of $3052.49 after 5 years. What annual interest rate on a certificate of deposit compounded monthly would yield an equivalent return?

92. **Mutual Funds**  A sales representative for a mutual funds company describes a “guaranteed investment fund” that the company is offering to new investors. You are told that if you deposit $10,000 in the fund you will be guaranteed a company describes a “guaranteed investment fund” that the company is offering to new investors. You are told that if you deposit $10,000 in the fund you will be guaranteed a return of at least $25,000 after 20 years. (Assume the interest is compounded quarterly.)

(a) What is the annual interest rate if the investment only meets the minimum guaranteed amount?

(b) After 20 years, you receive $32,000. What is the annual interest rate?

93. **Number of Doctors**  The number of medical doctors \( D \) (in thousands) in the United States from 1994 to 2002 can be modeled by

\[ D = 463.97 + 111.6\sqrt{t}, \quad 4 \leq t \leq 12 \]

where \( t \) represents the year, with \( t = 4 \) corresponding to 1994.  (Source: American Medical Association)

(a) In which year did the number of medical doctors reach 816,000?

(b) Use the model to predict when the number of medical doctors will reach 900,000. Is this prediction reasonable? Explain.
94. **Voting Population** The total voting-age population \( P \) (in millions) in the United States from 1990 to 2002 can be modeled by

\[
P = \frac{182.45 - 3.189t}{1.00 - 0.026r}, \quad 0 \leq t \leq 12
\]

where \( t \) represents the year, with \( t = 0 \) corresponding to 1990. (Source: U.S. Census Bureau)

(a) In which year did the total voting-age population reach 200 million?
(b) Use the model to predict when the total voting-age population will reach 230 million. Is this prediction reasonable? Explain.

95. **Saturated Steam** The temperature \( T \) (in degrees Fahrenheit) of saturated steam increases as pressure increases. This relationship is approximated by the model

\[
T = 75.82 - 2.11x + 43.51\sqrt{x}, \quad 5 \leq x \leq 40
\]

where \( x \) is the absolute pressure (in pounds per square inch).

(a) Use the model to complete the table.

<table>
<thead>
<tr>
<th>Absolute pressure, ( x )</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature, ( T )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) The temperature of steam at sea level is 212°F. Use the table in part (a) to approximate the absolute pressure at this temperature.
(c) Solve part (b) algebraically.
(d) Use a graphing utility to verify your solutions from parts (b) and (c).

96. **Airline Passengers** An airline offers daily flights between Chicago and Denver. The total monthly cost \( C \) (in millions of dollars) of these flights is

\[
C = \sqrt{0.2x + 1}
\]

where \( x \) is the number of passengers (in thousands). The total cost of the flights for June is 2.5 million dollars. How many passengers flew in June?

97. **Demand** The demand equation for a video game is modeled by

\[
p = 40 - \sqrt{0.01x + 1}
\]

where \( x \) is the number of units demanded per day and \( p \) is the price per unit. Approximate the demand when the price is $37.55.

98. **Demand** The demand equation for a high definition television set is modeled by

\[
p = 800 - \sqrt{0.01x + 1}
\]

where \( x \) is the number of units demanded per month and \( p \) is the price per unit. Approximate the demand when the price is $750.

99. **Baseball** A baseball diamond has the shape of a square in which the distance from home plate to second base is approximately \( 127\frac{1}{2} \) feet. Approximate the distance between the bases.

100. **Meteorology** A meteorologist is positioned 100 feet from the point where a weather balloon is launched. When the balloon is at height \( h \), the distance \( d \) (in feet) between the meteorologist and the balloon is

\[
d = \sqrt{100^2 + h^2}
\]

(a) Use a graphing utility to graph the equation. Use the \( \text{trace} \) feature to approximate the value of \( h \) when \( d = 200 \).
(b) Complete the table. Use the table to approximate the value of \( h \) when \( d = 200 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>160</th>
<th>165</th>
<th>170</th>
<th>175</th>
<th>180</th>
<th>185</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Find \( h \) algebraically when \( d = 200 \).
(d) Compare the results of each method. In each case, what information did you gain that wasn’t apparent in another solution method?

101. **Geometry** You construct a cone with a base radius of 8 inches. The surface area \( S \) of the cone can be represented by the equation

\[
S = 8\pi\sqrt{64 + h^2}
\]

where \( h \) is the height of the cone.

(a) Use a graphing utility to graph the equation. Use the \( \text{trace} \) feature to approximate the value of \( h \) when \( S = 350 \) square inches.
(b) Complete the table. Use the table to approximate the value of \( h \) when \( S = 350 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Find \( h \) algebraically when \( S = 350 \).
(d) Compare the results of each method. In each case, what information did you gain that wasn’t apparent in another solution method?
102. Labor  Working together, two people can complete a task in 8 hours. Working alone, one person takes 2 hours longer than the other to complete the task. How long would it take for each person to complete the task?

103. Labor  Working together, two people can complete a task in 12 hours. Working alone, one person takes 3 hours longer than the other to complete the task. How long would it take for each person to complete the task?

Model It

104. Power Line  A power station is on one side of a river that is 1/2 mile wide, and a factory is 8 miles downstream on the other side of the river, as shown in the figure. It costs $24 per foot to run power lines over land and $30 per foot to run them under water.
(a) Write the total cost C to run power lines in terms of x (see figure).
(b) Find the total cost when x = 3.
(c) Find the length x when C = $1,098,662.40.
(d) Use a graphing utility to graph the equation from part (a).
(e) Use your graph from part (d) to find the value of x that minimizes the cost.

105. A Person’s Tangential Speed in a Rotor
Solve for g: \( v = \sqrt{\frac{gR}{\mu s}} \)

106. Inductance
Solve for Q: \( i = \pm \sqrt{\frac{1}{LC} \sqrt{Q^2 - q}} \)

Synthesis

True or False? In Exercises 107 and 108, determine whether the statement is true or false. Justify your answer.

107. An equation can never have more than one extraneous solution.

108. When solving an absolute value equation, you will always have to check more than one solution.

In Exercises 109 and 110, find x such that the distance between the given points is 13. Explain your results.

109. (1, 2), (x, -10)
110. (-8, 0), (x, 5)

In Exercises 111 and 112, find y such that the distance between the given points is 17. Explain your results.

111. (0, 0), (8, y)
112. (-8, 4), (7, y)

In Exercises 113 and 114, consider an equation of the form \( x + |x - a| = b \), where a and b are constants.

113. Find a and b when the solution of the equation is \( x = 9 \). (There are many correct answers.)
114. Writing  Write a short paragraph listing the steps required to solve this equation involving absolute values and explain why it is important to check your solutions.

In Exercises 115 and 116, consider an equation of the form \( x + \sqrt{x - a} - b \), where a and b are constants.

115. Find a and b when the solution of the equation is \( x = 20 \). (There are many correct answers.)
116. Writing  Write a short paragraph listing the steps required to solve this equation involving radicals and explain why it is important to check your solutions.

Skills Review

In Exercises 117–120, perform the operation and simplify.

117. \( \frac{8}{3x} + \frac{3}{2x} \)
118. \( \frac{2}{x^2 - 4} - \frac{1}{x^2 - 3x + 2} \)
119. \( \frac{2}{z + 2} - \left( \frac{3 - \frac{2}{z}}{z} \right) \)
120. \( 25y^2 \div \frac{xy}{5} \)

In Exercises 121 and 122, find all real solutions of the equation.

121. \( x^2 - 22x + 121 = 0 \)
122. \( x(x - 20) + 3(x - 20) = 0 \)
1.7 Linear Inequalities in One Variable

What you should learn
- Represent solutions of linear inequalities in one variable.
- Solve linear inequalities in one variable.
- Solve inequalities involving absolute values.
- Use inequalities to model and solve real-life problems.

Why you should learn it
Inequalities can be used to model and solve real-life problems. For instance, in Exercise 101 on page 152, you will use a linear inequality to analyze the average salary for elementary school teachers.

Introduction
Simple inequalities were reviewed in Section P.1. There, you used the inequality symbols <, ≤, >, and ≥ to compare two numbers and to denote subsets of real numbers. For instance, the simple inequality

\[ x \geq 3 \]

denotes all real numbers \( x \) that are greater than or equal to 3.

In this section, you will expand your work with inequalities to include more involved statements such as

\[ 5x - 7 < 3x + 9 \]

and

\[ -3 \leq 6x - 1 < 3. \]

As with an equation, you solve an inequality in the variable \( x \) by finding all values of \( x \) for which the inequality is true. Such values are solutions and are said to satisfy the inequality. The set of all real numbers that are solutions of an inequality is the solution set of the inequality. For instance, the solution set of

\[ x + 1 < 4 \]

is all real numbers that are less than 3.

The set of all points on the real number line that represent the solution set is the graph of the inequality. Graphs of many types of inequalities consist of intervals on the real number line. See Section P.1 to review the nine basic types of intervals on the real number line. Note that each type of interval can be classified as bounded or unbounded.

Example 1  Intervals and Inequalities

Write an inequality to represent each interval, and state whether the interval is bounded or unbounded.

a. \((-3, 5]\)
b. \((-3, \infty)\)
c. \([0, 2]\)
d. \((-\infty, \infty)\)

Solution

a. \((-3, 5]\) corresponds to \(-3 < x \leq 5\). Bounded
b. \((-3, \infty)\) corresponds to \(-3 < x\). Unbounded
c. \([0, 2]\) corresponds to \(0 \leq x \leq 2\). Bounded
d. \((-\infty, \infty)\) corresponds to \(-\infty < x < \infty\). Unbounded

CHECKPOINT  Now try Exercise 1.
Properties of Inequalities

The procedures for solving linear inequalities in one variable are much like those for solving linear equations. To isolate the variable, you can make use of the Properties of Inequalities. These properties are similar to the properties of equality, but there are two important exceptions. When each side of an inequality is multiplied or divided by a negative number, the direction of the inequality symbol must be reversed. Here is an example.

\[-2 < 5 \quad \text{Original inequality}\]

\[(-3)(-2) > (-3)(5) \quad \text{Multiply each side by } -3 \text{ and reverse inequality.}\]

\[6 > -15 \quad \text{Simplify.}\]

Notice that if the inequality was not reversed you would obtain the false statement \(6 < -15\).

Two inequalities that have the same solution set are equivalent. For instance, the inequalities

\[x + 2 < 5\]

and

\[x < 3\]

are equivalent. To obtain the second inequality from the first, you can subtract 2 from each side of the inequality. The following list describes the operations that can be used to create equivalent inequalities.

**Properties of Inequalities**

Let \(a, b, c,\) and \(d\) be real numbers.

1. Transitive Property

   \[a < b \text{ and } b < c \quad \Rightarrow \quad a < c\]

2. Addition of Inequalities

   \[a < b \text{ and } c < d \quad \Rightarrow \quad a + c < b + d\]

3. Addition of a Constant

   \[a < b \quad \Rightarrow \quad a + c < b + c\]

4. Multiplication by a Constant

   For \(c > 0, a < b \quad \Rightarrow \quad ac < bc\)

   For \(c < 0, a < b \quad \Rightarrow \quad ac > bc \quad \text{Reverse the inequality.}\)

Each of the properties above is true if the symbol \(<\) is replaced by \(\leq\) and the symbol \(>\) is replaced by \(\geq\). For instance, another form of the multiplication property would be as follows.

For \(c > 0, a \leq b \quad \Rightarrow \quad ac \leq bc\)

For \(c < 0, a \leq b \quad \Rightarrow \quad ac \geq bc\)
Solving a Linear Inequality in One Variable

The simplest type of inequality is a linear inequality in one variable. For instance, $2x + 3 > 4$ is a linear inequality in $x$.

In the following examples, pay special attention to the steps in which the inequality symbol is reversed. Remember that when you multiply or divide by a negative number, you must reverse the inequality symbol.

**Example 2** Solving Linear Inequalities

Solve each inequality.

**a.** $5x - 7 > 3x + 9$

**Solution**

Write original inequality.

$2x - 7 > 9$

Subtract $3x$ from each side.

$2x > 16$

Add 7 to each side.

$x > 8$

Divide each side by 2.

The solution set is all real numbers that are greater than 8, which is denoted by $(8, \infty)$. The graph of this solution set is shown in Figure 1.23. Note that a parenthesis at 8 on the real number line indicates that 8 is not part of the solution set.

![Figure 1.23](https://example.com/figure123.png)

**Solution interval:** $(8, \infty)$

**b.** $1 - \frac{3x}{2} \geq x - 4$

Write original inequality.

$2 - 3x \geq 2x - 8$

Multiply each side by 2.

$2 - 5x \geq -8$

Subtract $2x$ from each side.

$-5x \geq -10$

Subtract 2 from each side.

$x \leq 2$

Divide each side by $-5$ and reverse the inequality.

The solution set is all real numbers that are less than or equal to 2, which is denoted by $(-\infty, 2]$. The graph of this solution set is shown in Figure 1.24. Note that a bracket at 2 on the real number line indicates that 2 is part of the solution set.

![Figure 1.24](https://example.com/figure124.png)

**Solution interval:** $(-\infty, 2]$

Now try Exercise 25.
Sometimes it is possible to write two inequalities as a double inequality. For instance, you can write the two inequalities \(-4 \leq 5x - 2\) and \(5x - 2 < 7\) more simply as

\[-4 \leq 5x - 2 < 7.\]  

This form allows you to solve the two inequalities together, as demonstrated in Example 3.

**Example 3**  
**Solving a Double Inequality**

To solve a double inequality, you can isolate \(x\) as the middle term.

\[
\begin{align*}
-3 &\leq 6x - 1 < 3 & \text{Original inequality} \\
-3 + 1 &\leq 6x - 1 + 1 < 3 + 1 & \text{Add 1 to each part.} \\
-2 &\leq 6x < 4 & \text{Simplify.} \\
\frac{-2}{6} &\leq \frac{6x}{6} < \frac{4}{6} & \text{Divide each part by 6.} \\
\frac{-1}{3} &\leq x < \frac{2}{3} & \text{Simplify.}
\end{align*}
\]

The solution set is all real numbers that are greater than or equal to \(-\frac{1}{3}\) and less than \(\frac{2}{3}\), which is denoted by \([-\frac{1}{3}, \frac{2}{3})\). The graph of this solution set is shown in Figure 1.25.

![Figure 1.25](image)

Solution interval: \([-\frac{1}{3}, \frac{2}{3})\)

Now try Exercise 37.

The double inequality in Example 3 could have been solved in two parts as follows.

\[
\begin{align*}
-3 &\leq 6x - 1 & \text{ and } & 6x - 1 < 3 \\
-2 &\leq 6x & \text{ and } & 6x < 3 \\
\frac{-1}{3} &\leq x & \text{ and } & x < \frac{2}{3}
\end{align*}
\]

The solution set consists of all real numbers that satisfy both inequalities. In other words, the solution set is the set of all values of \(x\) for which

\[-\frac{1}{3} \leq x < \frac{2}{3}.
\]

When combining two inequalities to form a double inequality, be sure that the inequalities satisfy the Transitive Property. For instance, it is incorrect to combine the inequalities \(3 < x\) and \(x \leq -1\) as \(3 < x \leq -1\). This “inequality” is wrong because 3 is not less than \(-1\).
Inequalities Involving Absolute Values

Solving an Absolute Value Inequality

Let x be a variable or an algebraic expression and let a be a real number such that a ≥ 0.

1. The solutions of |x| < a are all values of x that lie between −a and a.
   \[ |x| < a \quad \text{if and only if} \quad -a < x < a. \]
   Double inequality

2. The solutions of |x| > a are all values of x that are less than −a or greater than a.
   \[ |x| > a \quad \text{if and only if} \quad x < -a \quad \text{or} \quad x > a. \]
   Compound inequality

These rules are also valid if < is replaced by ≤ and > is replaced by ≥.

Example 4

Solving an Absolute Value Inequality

Solve each inequality.

a. \( |x - 5| < 2 \)

\[ x - 5 < 2 \quad \text{and} \quad x - 5 > -2 \]
\[ x < 7 \quad \text{and} \quad x > 3 \]
\[ 3 < x < 7 \]

The solution set is all real numbers that are greater than 3 and less than 7, which is denoted by (3, 7). The graph of this solution set is shown in Figure 1.26.

b. \( |x + 3| \geq 7 \)

\[ x + 3 \leq -7 \quad \text{or} \quad x + 3 \geq 7 \]
\[ x \leq -10 \quad \text{or} \quad x \geq 4 \]

The solution set is all real numbers that are less than or equal to −10 or greater than or equal to 4. The interval notation for this solution set is \((-\infty, -10] \cup [4, \infty)\). The symbol \(\cup\) is called a union symbol and is used to denote the combining of two sets. The graph of this solution set is shown in Figure 1.27.

Now try Exercise 49.
Applications

The problem-solving plan described in Section 1.3 can be used to model and solve real-life problems that involve inequalities, as illustrated in Example 5.

Example 5  Comparative Shopping

You are choosing between two different cell phone plans. Plan A costs $49.99 per month for 500 minutes plus $0.40 for each additional minute. Plan B costs $45.99 per month for 500 minutes plus $0.45 for each additional minute. How many additional minutes must you use in one month for plan B to cost more than plan A?

Solution

Verbal Model:

<table>
<thead>
<tr>
<th>Monthly cost for plan B</th>
<th>Monthly cost for plan A</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.45m + 45.99</td>
<td>0.40m + 49.99</td>
</tr>
</tbody>
</table>

Labels: Minutes used (over 500) in one month = \( m \) (minutes)

Monthly cost for plan A = \( 0.40m + 49.99 \) (dollars)

Monthly cost for plan B = \( 0.45m + 45.99 \) (dollars)

Inequality: \( 0.45m + 45.99 > 0.40m + 49.99 \)

\[ 0.05m > 4 \]

\[ m > 80 \text{ minutes} \]

Plan B costs more if you use more than 80 additional minutes in one month.

Now try Exercise 91.

Example 6  Accuracy of a Measurement

You go to a candy store to buy chocolates that cost $9.89 per pound. The scale that is used in the store has a state seal of approval that indicates the scale is accurate to within half an ounce (or \( \frac{1}{32} \) of a pound). According to the scale, your purchase weighs one-half pound and costs $4.95. How much might you have been undercharged or overcharged as a result of inaccuracy in the scale?

Solution

Let \( x \) represent the true weight of the candy. Because the scale is accurate to within half an ounce (or \( \frac{1}{32} \) of a pound), the difference between the exact weight (\( x \)) and the scale weight (\( \frac{1}{2} \)) is less than or equal to \( \frac{1}{32} \) of a pound. That is, \( |x - \frac{1}{2}| \leq \frac{1}{32} \). You can solve this inequality as follows.

\[ -\frac{1}{32} \leq x - \frac{1}{2} \leq \frac{1}{32} \]

\[ \frac{15}{32} \leq x \leq \frac{17}{32} \]

\[ 0.46875 \leq x \leq 0.53125 \]

In other words, your “one-half pound” of candy could have weighed as little as 0.46875 pound (which would have cost $4.64) or as much as 0.53125 pound (which would have cost $5.25). So, you could have been overcharged by as much as $0.31 or undercharged by as much as $0.30.

Now try Exercise 105.
1.7 Exercises

VOCABULARY CHECK: Fill in the blanks.
1. The set of all real numbers that are solutions to an inequality is the ________ of the inequality.
2. The set of all points on the real number line that represent the solution set of an inequality is the ________ of the inequality.
3. To solve a linear inequality in one variable, you can use the properties of inequalities, which are identical to those used to solve equations, with the exception of multiplying or dividing each side by a ________ number.
4. Two inequalities that have the same solution set are ________ ________.
5. It is sometimes possible to write two inequalities as one inequality, called a ________ inequality.
6. The symbol \( \cup \) is called a ________ symbol and is used to denote the combining of two sets.


In Exercises 1–6, (a) write an inequality that represents the interval and (b) state whether the interval is bounded or unbounded.

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>15. ( 0 &lt; \frac{x - 2}{4} &lt; 2 )</td>
<td>(a) ( x = 4 ) (b) ( x = 10 )</td>
</tr>
<tr>
<td>16. ( -1 &lt; \frac{3 - x}{2} \leq 1 )</td>
<td>(a) ( x = 0 ) (b) ( x = -5 )</td>
</tr>
<tr>
<td>17. (</td>
<td>x - 10</td>
</tr>
<tr>
<td>18. (</td>
<td>2x - 3</td>
</tr>
</tbody>
</table>

In Exercises 19–44, solve the inequality and sketch the solution on the real number line. (Some inequalities have no solutions.)

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>19. ( 4x &lt; 12 )</td>
<td></td>
</tr>
<tr>
<td>20. ( 10x &lt; -40 )</td>
<td></td>
</tr>
<tr>
<td>21. ( -2x &gt; -3 )</td>
<td></td>
</tr>
<tr>
<td>22. ( -6x &gt; 15 )</td>
<td></td>
</tr>
<tr>
<td>23. ( x - 5 \geq 7 )</td>
<td></td>
</tr>
<tr>
<td>24. ( x + 7 \leq 12 )</td>
<td></td>
</tr>
<tr>
<td>25. ( 2x + 7 &lt; 3 + 4x )</td>
<td></td>
</tr>
<tr>
<td>26. ( 3x + 1 \geq 2 + x )</td>
<td></td>
</tr>
<tr>
<td>27. ( 2x - 1 \geq 1 - 5x )</td>
<td></td>
</tr>
<tr>
<td>28. ( 6x - 4 \leq 2 + 8x )</td>
<td></td>
</tr>
<tr>
<td>29. ( 4 - 2x &lt; 3(3 - x) )</td>
<td></td>
</tr>
<tr>
<td>30. ( 4(x + 1) &lt; 2x + 3 )</td>
<td></td>
</tr>
<tr>
<td>31. ( \frac{2}{3}x - 6 \leq x - 7 )</td>
<td></td>
</tr>
<tr>
<td>32. ( 3 + \frac{2}{5}x &gt; x - 2 )</td>
<td></td>
</tr>
<tr>
<td>33. ( \frac{3}{4}(8x + 1) \geq 3x + \frac{5}{2} )</td>
<td></td>
</tr>
<tr>
<td>34. ( 9x - 1 &lt; \frac{1}{3}(16x - 2) )</td>
<td></td>
</tr>
<tr>
<td>35. ( 3.6x + 11 \geq -3.4 )</td>
<td></td>
</tr>
<tr>
<td>36. ( 15.6 - 1.3x &lt; -5.2 )</td>
<td></td>
</tr>
<tr>
<td>37. ( 1 &lt; 2x + 3 &lt; 9 )</td>
<td></td>
</tr>
<tr>
<td>38. ( -8 \leq -3(x + 5) &lt; 13 )</td>
<td></td>
</tr>
<tr>
<td>39. ( -4 &lt; \frac{2x - 3}{3} &lt; 4 )</td>
<td></td>
</tr>
<tr>
<td>40. ( 0 \leq \frac{x + 3}{2} &lt; 5 )</td>
<td></td>
</tr>
<tr>
<td>41. ( \frac{3}{5} &gt; x + 1 \geq \frac{1}{4} )</td>
<td></td>
</tr>
<tr>
<td>42. ( -1 &lt; 2 - \frac{x}{3} &lt; 1 )</td>
<td></td>
</tr>
<tr>
<td>43. ( 3.2 \leq 0.4x - 1 \leq 4.4 )</td>
<td></td>
</tr>
<tr>
<td>44. ( 4.5 &gt; \frac{1.5x + 6}{2} &gt; 10.5 )</td>
<td></td>
</tr>
</tbody>
</table>
In Exercises 45–60, solve the inequality and sketch the solution on the real number line. (Some inequalities have no solution.)

45. \(|x| < 6\)  
46. \(|x| > 4\) 
47. \(\frac{x}{2} > 1\)  
48. \(\frac{x}{5} > 3\) 
49. \(|x - 5| < -1\)  
50. \(|x - 7| < -5\) 
51. \(|x - 20| \leq 6\)  
52. \(|x - 8| \geq 0\) 
53. \(3 - 4x \geq 9\)  
54. \(|1 - 2x| < 5\) 
55. \(\frac{|x - 3|}{2} \geq 4\)  
56. \(1 - \frac{2x}{3} < 1\) 
57. \(9 - 2x > 2 < -1\)  
58. \(|x + 14| + 3 > 17\) 
59. \(2|x + 10| \geq 9\)  
60. \(3|x - 5x| \leq 9\)

\textbf{Graphical Analysis} In Exercises 61–68, use a graphing utility to graph the inequality and identify the solution set.

61. \(6x > 12\)  
62. \(3x - 1 \leq 5\) 
63. \(5 - 2x \geq 1\)  
64. \(3(x + 1) < x + 7\) 
65. \(|x - 8| \leq 14\)  
66. \(|2x + 9| > 13\) 
67. \(2|x + 7| \geq 13\)  
68. \(\frac{1}{2}|x + 1| \leq 3\)

\textbf{Graphical Analysis} In Exercises 69–74, use a graphing utility to graph the equation. Use the graph to approximate the values of \(x\) that satisfy each inequality.

\begin{align*}
69. & \ y = 2x - 3 \\
& \text{Inequalities} \\
& (a) \ y \geq 1 \\
& (b) \ y \leq 0 \\
70. & \ y = \frac{2}{3}x + 1 \\
& (a) \ y \leq 5 \\
& (b) \ y \geq 0 \\
71. & \ y = -\frac{1}{2}x + 2 \\
& (a) \ 0 \leq y \leq 3 \\
& (b) \ y \geq 0 \\
72. & \ y = -3x + 8 \\
& (a) \ -1 \leq y \leq 3 \\
& (b) \ y \leq 0 \\
73. & \ y = |x - 3| \\
& (a) \ y \leq 2 \\
& (b) \ y \geq 4 \\
74. & \ y = \frac{1}{2}x + 1 \\
& (a) \ y \leq 4 \\
& (b) \ y \geq 1 \\
\end{align*}

In Exercises 75–80, find the interval(s) on the real number line for which the radicand is nonnegative.

75. \(\sqrt{x - 5}\)  
76. \(\sqrt{x - 10}\) 
77. \(\sqrt{x + 3}\)  
78. \(\sqrt{3 - x}\) 
79. \(\sqrt{7 - 2x}\)  
80. \(\sqrt{6x + 15}\)

81. \textbf{Think About It} The graph of \(|x - 5| < 3\) can be described as all real numbers within three units of 5. Give a similar description of \(|x - 10| < 8\).

82. \textbf{Think About It} The graph of \(|x - 2| > 5\) can be described as all real numbers more than five units from 2. Give a similar description of \(|x - 8| > 4\).

In Exercises 83–90, use absolute value notation to define the interval (or pair of intervals) on the real number line.

83. 
84. 
85. 
86. 
87. All real numbers within 10 units of 12  
88. All real numbers at least five units from 8  
89. All real numbers more than four units from \(-3\)  
90. All real numbers no more than seven units from \(-6\)

91. \textbf{Checking Account} You can choose between two types of checking accounts at your local bank. Type A charges a monthly service fee of $6 plus $0.25 for each check written. Type B charges a monthly service fee of $4.50 plus $0.50 for each check written. How many checks must you write in a month in order for the monthly charges for type A to be less than that for type B?

92. \textbf{Copying Costs} Your department sends its copying to the photocopier center of your company. The center bills your department $0.10 per page. You have investigated the possibility of buying a departmental copier for $3000. With your own copier, the cost per page would be $0.03. The expected life of the copier is 4 years. How many copies must you make in the four-year period to justify buying the copier?

93. \textbf{Investment} In order for an investment of $1000 to grow to more than $1062.50 in 2 years, what must the annual interest rate be? \([A = P(1 + rt)]\)

94. \textbf{Investment} In order for an investment of $750 to grow to more than $825 in 2 years, what must the annual interest rate be? \([A = P(1 + rt)]\)

95. \textbf{Cost, Revenue, and Profit} The revenue for selling \(x\) units of a product is \(R = 115.95x\). The cost of producing \(x\) units is

\[C = 95x + 750.\]

To obtain a profit, the revenue must be greater than the cost. For what values of \(x\) will this product return a profit?

96. \textbf{Cost, Revenue, and Profit} The revenue for selling \(x\) units of a product is \(R = 24.55x\). The cost of producing \(x\) units is

\[C = 15.4x + 150,000.\]

To obtain a profit, the revenue must be greater than the cost. For what values of \(x\) will this product return a profit?

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97. **Daily Sales** A doughnut shop sells a dozen doughnuts for $2.95. Beyond the fixed costs (rent, utilities, and insurance) of $150 per day, it costs $1.45 for enough materials (flour, sugar, and so on) and labor to produce a dozen doughnuts. The daily profit from doughnut sales varies between $50 and $200. Between what levels (in dozens) do the daily sales vary?

98. **Weight Loss Program** A person enrolls in a diet and exercise program that guarantees a loss of at least pounds per week. The person’s weight at the beginning of the program is 164 pounds. Find the maximum number of weeks before the person attains a goal weight of 128 pounds.

99. **Data Analysis: IQ Scores and GPA** The admissions office of a college wants to determine whether there is a relationship between IQ scores $x$ and grade-point averages $y$ after the first year of school. An equation that models the data the admissions office obtained is $y = 0.067x - 5.638$.

(a) Use a graphing utility to graph the model.

(b) Use the graph to estimate the values of $x$ that predict a grade-point average of at least 3.0.

100. **Data Analysis: Weightlifting** You want to determine whether there is a relationship between an athlete’s weight $x$ (in pounds) and the athlete’s maximum bench-press weight $y$ (in pounds). The table shows a sample of data from 12 athletes.

<table>
<thead>
<tr>
<th>Athlete’s weight, $x$</th>
<th>Bench-press weight, $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>165</td>
<td>170</td>
</tr>
<tr>
<td>184</td>
<td>185</td>
</tr>
<tr>
<td>150</td>
<td>200</td>
</tr>
<tr>
<td>210</td>
<td>255</td>
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<tr>
<td>196</td>
<td>205</td>
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<td>240</td>
<td>295</td>
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<td>190</td>
<td>185</td>
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<tr>
<td>230</td>
<td>250</td>
</tr>
<tr>
<td>160</td>
<td>155</td>
</tr>
</tbody>
</table>

(a) Use a graphing utility to plot the data.

(b) A model for the data is $y = 1.3x - 36$. Use a graphing utility to graph the model in the same viewing window used in part (a).

(c) Use the graph to estimate the values of $x$ that predict a maximum bench-press weight of at least 200 pounds.

(d) Verify your estimate from part (c) algebraically.

(e) Use the graph to write a statement about the accuracy of the model. If you think the graph indicates that an athlete’s weight is not a particularly good indicator of the athlete’s maximum bench-press weight, list other factors that might influence an individual’s maximum bench-press weight.

101. **Teachers’ Salaries** The average salary $S$ (in thousands of dollars) for elementary school teachers in the United States from 1990 to 2002 is approximated by the model $S = 1.05t + 31.0$, $0 \leq t \leq 12$ where $t$ represents the year, with $t = 0$ corresponding to 1990. (Source: National Education Association)

(a) According to this model, when was the average salary at least $32,000, but not more than $42,000?

(b) According to this model, when will the average salary exceed $48,000?

102. **Egg Production** The number of eggs $E$ (in billions) produced in the United States from 1990 to 2002 can be modeled by $E = 1.64t + 67.2$, $0 \leq t \leq 12$ where $t$ represents the year, with $t = 0$ corresponding to 1990. (Source: U.S. Department of Agriculture)

(a) According to this model, when was the annual egg production 70 billion, but no more than 80 billion?

(b) According to this model, when will the annual egg production exceed 95 billion?

103. **Geometry** The side of a square is measured as 10.4 inches with a possible error of $\frac{1}{10}$ inch. Using these measurements, determine the interval containing the possible areas of the square.

104. **Geometry** The side of a square is measured as 24.2 centimeters with a possible error of 0.25 centimeter. Using these measurements, determine the interval containing the possible areas of the square.

105. **Accuracy of Measurement** You stop at a self-service gas station to buy 15 gallons of 87-octane gasoline at $1.89 a gallon. The gas pump is accurate to within $\frac{1}{10}$ gallon of a gallon. How much might you be undercharged or overcharged?
106. **Accuracy of Measurement** You buy six T-bone steaks that cost $14.99 per pound. The weight that is listed on the package is 5.72 pounds. The scale that weighed the package is accurate to within $\frac{1}{2}$ ounce. How much might you be undercharged or overcharged?

107. **Time Study** A time study was conducted to determine the length of time required to perform a particular task in a manufacturing process. The times required by approximately two-thirds of the workers in the study satisfied the inequality

$$\left| \frac{t - 15.6}{1.9} \right| < 1$$

where $t$ is time in minutes. Determine the interval on the real number line in which these times lie.

108. **Height** The heights $h$ of two-thirds of the members of a population satisfy the inequality

$$\left| \frac{h - 68.5}{2.7} \right| \leq 1$$

where $h$ is measured in inches. Determine the interval on the real number line in which these heights lie.

109. **Meteorology** An electronic device is to be operated in an environment with relative humidity $h$ in the interval defined by $|h - 50| \leq 30$. What are the minimum and maximum relative humidities for the operation of this device?

110. **Music** Michael Kasha of Florida State University used physics and mathematics to design a new classical guitar. He used the model for the frequency of the vibrations on a circular plate

$$v = \frac{2.6t}{d^2} \sqrt{\frac{E}{\rho}}$$

where $v$ is the frequency (in vibrations per second), $t$ is the plate thickness (in millimeters), $d$ is the diameter of the plate, $E$ is the elasticity of the plate material, and $\rho$ is the density of the plate material. For fixed values of $d$, $E$, and $\rho$, the graph of the equation is a line (see figure).

(b) Estimate the plate thickness when the frequency is 600 vibrations per second.

(c) Approximate the interval for the plate thickness when the frequency is between 200 and 400 vibrations per second.

(d) Approximate the interval for the frequency when the plate thickness is less than 3 millimeters.

**Synthesis**

**True or False?** In Exercises 111 and 112, determine whether the statement is true or false. Justify your answer.

111. If $a$, $b$, and $c$ are real numbers, and $a \leq b$, then $ac \leq bc$.

112. If $-10 \leq x \leq 8$, then $-10 \geq -x$ and $-x \geq -8$.

113. Identify the graph of the inequality $|x - a| \geq 2$.

(a) \[ \begin{array}{c|c|c} \hline a-2 & a & a+2 \hline \end{array} \]

(b) \[ \begin{array}{c|c|c} \hline a-2 & a & a+2 \hline \end{array} \]

(c) \[ \begin{array}{c|c|c} \hline 2-a & 2 & 2+a \hline \end{array} \]

(d) \[ \begin{array}{c|c|c} \hline 2-a & 2 & 2+a \hline \end{array} \]

114. Find sets of values of $a$, $b$, and $c$ such that $0 \leq x \leq 10$ is a solution of the inequality $|ax - b| \leq c$.

**Skills Review**

In Exercises 115–118, find the distance between each pair of points. Then find the midpoint of the line segment joining the points.

115. $(-4, 2), (1, 12)$

116. $(1, -2), (10, 3)$

117. $(3, 6), (-5, -8)$

118. $(0, -3), (-6, 9)$

In Exercises 119–122, solve the equation.

119. $-6(2 - x) - 12 = 36$

120. $4(x + 7) - 9 = -6(-x - 1)$

121. $14x^2 + 5x - 1 = 0$

122. $x^3 + 5x^2 - 4x - 20 = 0$

123. Find the coordinates of the point located 3 units to the left of the $y$-axis and 10 units above the $x$-axis.

124. Determine the quadrant(s) in which the point $(x, y)$ could be located if $y > 0$.

125. **Make a Decision** To work an extended application analyzing the number of heart disease deaths per 100,000 people in the United States, visit this text’s website at college.hmco.com. (Data Source: U.S. National Center for Health Statistics)
### Polynomial Inequalities

To solve a polynomial inequality such as \( x^2 - 2x - 3 < 0 \), you can use the fact that a polynomial can change signs only at its zeros (the \( x \)-values that make the polynomial equal to zero). Between two consecutive zeros, a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into intervals in which the polynomial has no sign changes. These zeros are the critical numbers of the inequality, and the resulting intervals are the test intervals for the inequality. For instance, the polynomial above factors as

\[
x^2 - 2x - 3 = (x + 1)(x - 3)
\]

and has two zeros, \( x = -1 \) and \( x = 3 \). These zeros divide the real number line into three test intervals:

\[
(−∞, −1), \quad (−1, 3), \quad \text{and} \quad (3, ∞).
\]

(See Figure 1.28.)

So, to solve the inequality \( x^2 - 2x - 3 < 0 \), you need only test one value from each of these test intervals to determine whether the value satisfies the original inequality. If so, you can conclude that the interval is a solution of the inequality.

![Figure 1.28](image.png)

**Three test intervals for** \( x^2 - 2x - 3 \)

You can use the same basic approach to determine the test intervals for any polynomial.

### Finding Test Intervals for a Polynomial

To determine the intervals on which the values of a polynomial are entirely negative or entirely positive, use the following steps.

1. Find all real zeros of the polynomial, and arrange the zeros in increasing order (from smallest to largest). These zeros are the critical numbers of the polynomial.

2. Use the critical numbers of the polynomial to determine its test intervals.

3. Choose one representative \( x \)-value in each test interval and evaluate the polynomial at that value. If the value of the polynomial is negative, the polynomial will have negative values for every \( x \)-value in the interval. If the value of the polynomial is positive, the polynomial will have positive values for every \( x \)-value in the interval.
Example 1  Solving a Polynomial Inequality

Solve
\[ x^2 - x - 6 < 0. \]

Solution

By factoring the polynomial as
\[ x^2 - x - 6 = (x + 2)(x - 3) \]
you can see that the critical numbers are \( x = -2 \) and \( x = 3 \). So, the polynomial’s test intervals are
\[ (-\infty, -2), \quad (-2, 3), \quad \text{and} \quad (3, \infty). \]

In each test interval, choose a representative \( x \)-value and evaluate the polynomial.

<table>
<thead>
<tr>
<th>Test Interval</th>
<th>( x )-Value</th>
<th>Polynomial Value</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, -2))</td>
<td>( x = -3 )</td>
<td>((-3)^2 - (-3) - 6 = 6)</td>
<td>Positive</td>
</tr>
<tr>
<td>((-2, 3))</td>
<td>( x = 0 )</td>
<td>((0)^2 - (0) - 6 = -6)</td>
<td>Negative</td>
</tr>
<tr>
<td>((3, \infty))</td>
<td>( x = 4 )</td>
<td>((4)^2 - (4) - 6 = 6)</td>
<td>Positive</td>
</tr>
</tbody>
</table>

From this you can conclude that the inequality is satisfied for all \( x \)-values in \((-2, 3)\). This implies that the solution of the inequality \( x^2 - x - 6 < 0 \) is the interval \((-2, 3)\), as shown in Figure 1.29. Note that the original inequality contains a less than symbol. This means that the solution set does not contain the endpoints of the test interval \((-2, 3)\).

Choose \( x = -3 \). \((x + 2)(x - 3) > 0\)

Choose \( x = 4 \). \((x + 2)(x - 3) > 0\)

Choose \( x = 0 \). \((x + 2)(x - 3) < 0\)

FIGURE 1.29

Now try Exercise 13.

As with linear inequalities, you can check the reasonableness of a solution by substituting \( x \)-values into the original inequality. For instance, to check the solution found in Example 1, try substituting several \( x \)-values from the interval \((-2, 3)\) into the inequality
\[ x^2 - x - 6 < 0. \]

Regardless of which \( x \)-values you choose, the inequality should be satisfied.

You can also use a graph to check the result of Example 1. Sketch the graph of \( y = x^2 - x - 6 \), as shown in Figure 1.30. Notice that the graph is below the \( x \)-axis on the interval \((-2, 3)\).
In Example 1, the polynomial inequality was given in general form (with the polynomial on one side and zero on the other). Whenever this is not the case, you should begin the solution process by writing the inequality in general form.

**Example 2**  Solving a Polynomial Inequality

Solve $2x^3 - 3x^2 - 32x > -48$.

**Solution**

Begin by writing the inequality in general form.

\[
2x^3 - 3x^2 - 32x > -48 \quad \text{Write original inequality.}
\]

\[
2x^3 - 3x^2 - 32x + 48 > 0 \quad \text{Write in general form.}
\]

\[
(x - 4)(x + 4)(2x - 3) > 0 \quad \text{Factor.}
\]

The critical numbers are $x = -4$, $x = \frac{3}{2}$, and $x = 4$, and the test intervals are $(-\infty, -4), (-4, \frac{3}{2}), (\frac{3}{2}, 4)$, and $(4, \infty)$.

<table>
<thead>
<tr>
<th>Test Interval</th>
<th>x-Value</th>
<th>Polynomial Value</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, -4)$</td>
<td>$x = -5$</td>
<td>$2(-5)^3 - 3(-5)^2 - 32(-5) + 48$</td>
<td>Negative</td>
</tr>
<tr>
<td>$(-4, \frac{3}{2})$</td>
<td>$x = 0$</td>
<td>$2(0)^3 - 3(0)^2 - 32(0) + 48$</td>
<td>Positive</td>
</tr>
<tr>
<td>$\left(\frac{3}{2}, 4\right)$</td>
<td>$x = 2$</td>
<td>$2(2)^3 - 3(2)^2 - 32(2) + 48$</td>
<td>Negative</td>
</tr>
<tr>
<td>$(4, \infty)$</td>
<td>$x = 5$</td>
<td>$2(5)^3 - 3(5)^2 - 32(5) + 48$</td>
<td>Positive</td>
</tr>
</tbody>
</table>

From this you can conclude that the inequality is satisfied on the open intervals $(-4, \frac{3}{2})$ and $(4, \infty)$. Therefore, the solution set consists of all real numbers in the intervals $(-4, \frac{3}{2})$ and $(4, \infty)$, as shown in Figure 1.31.

Now try Exercise 21.

When solving a polynomial inequality, be sure you have accounted for the particular type of inequality symbol given in the inequality. For instance, in Example 2, note that the original inequality contained a “greater than” symbol and the solution consisted of two open intervals. If the original inequality had been

\[
2x^3 - 3x^2 - 32x \geq -48
\]

the solution would have consisted of the closed interval $[-4, \frac{3}{2}]$ and the interval $[4, \infty)$.
Each of the polynomial inequalities in Examples 1 and 2 has a solution set that consists of a single interval or the union of two intervals. When solving the exercises for this section, watch for unusual solution sets, as illustrated in Example 3.

## Example 3  Unusual Solution Sets

a. The solution set of the following inequality consists of the entire set of real numbers, \((-\infty, \infty)\). In other words, the value of the quadratic \(x^2 + 2x + 4\) is positive for every real value of \(x\).

\[ x^2 + 2x + 4 > 0 \]

b. The solution set of the following inequality consists of the single real number \(-1\), because the quadratic \(x^2 + 2x + 1\) has only one critical number, \(x = -1\), and it is the only value that satisfies the inequality.

\[ x^2 + 2x + 1 \leq 0 \]

c. The solution set of the following inequality is empty. In other words, the quadratic \(x^2 + 3x + 5\) is not less than zero for any value of \(x\).

\[ x^2 + 3x + 5 < 0 \]

d. The solution set of the following inequality consists of all real numbers except \(x = 2\). In interval notation, this solution set can be written as \((-\infty, 2) \cup (2, \infty)\).

\[ x^2 - 4x + 4 > 0 \]

**CHECKPOINT**  Now try Exercise 25.

## Exploration

You can use a graphing utility to verify the results in Example 3. For instance, the graph of \(y = x^2 + 2x + 4\) is shown below. Notice that the \(y\)-values are greater than 0 for all values of \(x\), as stated in Example 3(a).

Use the graphing utility to graph the following:

\[
\begin{align*}
y &= x^2 + 2x + 1 \\
y &= x^2 + 3x + 5 \\
y &= x^2 - 4x + 4
\end{align*}
\]

Explain how you can use the graphs to verify the results of parts (b), (c), and (d) of Example 3.

![Graph of a quadratic function](image)
Rational Inequalities

The concepts of critical numbers and test intervals can be extended to rational inequalities. To do this, use the fact that the value of a rational expression can change sign only at its zeros (the \( x \)-values for which its numerator is zero) and its undefined values (the \( x \)-values for which its denominator is zero). These two types of numbers make up the critical numbers of a rational inequality. When solving a rational inequality, begin by writing the inequality in general form with the rational expression on the left and zero on the right.

Example 4  Solving a Rational Inequality

Solve \( \frac{2x - 7}{x - 5} \leq 3 \).

Solution

\[
\frac{2x - 7}{x - 5} \leq 3
\]

Write original inequality.

\[
\frac{2x - 7}{x - 5} - 3 \leq 0
\]

Write in general form.

\[
\frac{2x - 7 - 3x + 15}{x - 5} \leq 0
\]

Find the LCD and add fractions.

\[
\frac{-x + 8}{x - 5} \leq 0
\]

Simplify.

Critical numbers: \( x = 5, x = 8 \)  Zeros and undefined values of rational expression

Test intervals: \((-\infty, 5), (5, 8), (8, \infty)\)

Test:

\[
\frac{-x + 8}{x - 5} \leq 0?
\]

After testing these intervals, as shown in Figure 1.32, you can see that the inequality is satisfied on the open intervals \((-\infty, 5)\) and \((8, \infty)\). Moreover, because \(( -x + 8)/(x - 5) = 0\) when \( x = 8 \), you can conclude that the solution set consists of all real numbers in the intervals \((-\infty, 5) \cup [8, \infty)\). (Be sure to use a closed interval to indicate that \( x \) can equal 8.)

Choose \( x = 6 \).

\[
\frac{-x + 8}{x - 5} > 0
\]

Choose \( x = 4 \).

\[
\frac{-x + 8}{x - 5} < 0
\]

Choose \( x = 9 \).

\[
\frac{-x + 8}{x - 5} < 0
\]

FIGURE 1.32

Now try Exercise 39.
Applications

One common application of inequalities comes from business and involves profit, revenue, and cost. The formula that relates these three quantities is

\[ \text{Profit} = \text{Revenue} - \text{Cost} \]

\[ P = R - C. \]

**Example 5** Increasing the Profit for a Product

The marketing department of a calculator manufacturer has determined that the demand for a new model of calculator is

\[ p = 100 - 0.00001x, \quad 0 \leq x \leq 10,000,000 \]

Demand equation

where \( p \) is the price per calculator (in dollars) and \( x \) represents the number of calculators sold. (If this model is accurate, no one would be willing to pay $100 for the calculator. At the other extreme, the company couldn’t sell more than 10 million calculators.) The revenue for selling \( x \) calculators is

\[ R = xp = x(100 - 0.00001x) \]

Revenue equation

as shown in Figure 1.33. The total cost of producing \( x \) calculators is $10 per calculator plus a development cost of $2,500,000. So, the total cost is

\[ C = 10x + 2,500,000. \]

Cost equation

What price should the company charge per calculator to obtain a profit of at least $190,000,000?

**Solution**

**Verbal Model:**

Profit = Revenue - Cost

**Equation:**

\[ P = R - C \]

\[ P = 100x - 0.00001x^2 - (10x + 2,500,000) \]

\[ P = -0.00001x^2 + 90x - 2,500,000 \]

To answer the question, solve the inequality

\[ P \geq 190,000,000 \]

\[ -0.00001x^2 + 90x - 2,500,000 \geq 190,000,000. \]

When you write the inequality in general form, find the critical numbers and the test intervals, and then test a value in each test interval, you can find the solution to be

\[ 3,500,000 \leq x \leq 5,500,000 \]

as shown in Figure 1.34. Substituting the \( x \)-values in the original price equation shows that prices of

\[ $45.00 \leq p \leq $65.00 \]

will yield a profit of at least $190,000,000.

**Checkpoint**

Now try Exercise 71.
Another common application of inequalities is finding the domain of an expression that involves a square root, as shown in Example 6.

**Example 6**  
**Finding the Domain of an Expression**

Find the domain of \( \sqrt{64 - 4x^2} \).

**Algebraic Solution**

Remember that the domain of an expression is the set of all \( x \)-values for which the expression is defined. Because \( \sqrt{64 - 4x^2} \) is defined (has real values) only if \( 64 - 4x^2 \) is nonnegative, the domain is given by \( 64 - 4x^2 \geq 0 \).

\[
64 - 4x^2 \geq 0 \\
16 - x^2 \geq 0 \\
(4 - x)(4 + x) \geq 0
\]

Write in general form. 
Divide each side by 4.
Write in factored form.

So, the inequality has two critical numbers: \( x = -4 \) and \( x = 4 \). You can use these two numbers to test the inequality as follows.

**Critical numbers:** \( x = -4, x = 4 \)

**Test intervals:** \( (-\infty, -4), (-4, 4), (4, \infty) \)

**Test:** For what values of \( x \) is \( \sqrt{64 - 4x^2} \geq 0 \)?

A test shows that the inequality is satisfied in the closed interval \( [-4, 4] \). So, the domain of the expression \( \sqrt{64 - 4x^2} \) is the interval \( [-4, 4] \).

**Graphical Solution**

Begin by sketching the graph of the equation \( y = \sqrt{64 - 4x^2} \), as shown in Figure 1.35. From the graph, you can determine that the \( x \)-values extend from \(-4 \) to \( 4 \) (including \(-4 \) and \( 4 \)). So, the domain of the expression \( \sqrt{64 - 4x^2} \) is the interval \( [-4, 4] \).

![Graph of \( y = \sqrt{64 - 4x^2} \)](image)

To analyze a test interval, choose a representative \( x \)-value in the interval and evaluate the expression at that value. For instance, in Example 6, if you substitute any number from the interval \( [-4, 4] \) into the expression \( \sqrt{64 - 4x^2} \) you will obtain a nonnegative number under the radical symbol that simplifies to a real number. If you substitute any number from the intervals \( (-\infty, -4) \) and \( (4, \infty) \) you will obtain a complex number. It might be helpful to draw a visual representation of the intervals as shown in Figure 1.36.

![Interval representation](image)

### Writing About Mathematics

**Profit Analysis** Consider the relationship

\[
P = R - C
\]

described on page 159. Write a paragraph discussing why it might be beneficial to solve \( P < 0 \) if you owned a business. Use the situation described in Example 5 to illustrate your reasoning.
1.8 Exercises

VOCABULARY CHECK: Fill in the blanks.
1. To solve a polynomial inequality, find the ______ numbers of the polynomial, and use these numbers to create ______ for the inequality.
2. The critical numbers of a rational expression are its ______ and its ______.
3. The formula that relates cost, revenue, and profit is ______.


In Exercises 1–4, determine whether each value of $x$ is a solution of the inequality.

**Inequality** | **Values**
--- | ---
1. $x^2 - 3 < 0$ | (a) $x = 3$ (b) $x = 0$
(c) $x = \frac{3}{2}$ (d) $x = -5$
2. $x^2 - x - 12 \geq 0$ | (a) $x = 5$ (b) $x = 0$
(c) $x = -4$ (d) $x = -3$
3. $\frac{x + 2}{x - 4} \geq 3$ | (a) $x = 5$ (b) $x = 4$
(c) $x = -\frac{3}{2}$ (d) $x = \frac{9}{2}$
4. $\frac{3x^2}{x^2 + 4} < 1$ | (a) $x = -2$ (b) $x = -1$
(c) $x = 0$ (d) $x = 3$

In Exercises 5–8, find the critical numbers of the expression.

5. $2x^2 - x - 6$
6. $9x^3 - 25x^2$
7. $2 + \frac{3}{x - 5}$
8. $\frac{x}{x + 2} - \frac{2}{x - 1}$

In Exercises 9–26, solve the inequality and graph the solution on the real number line.

9. $x^2 \leq 9$
10. $x^2 < 36$
11. $(x + 2)^2 < 25$
12. $(x - 3)^2 \geq 1$
13. $x^2 + 4x + 4 \geq 9$
14. $x^2 - 6x + 9 < 16$
15. $x^2 + x < 6$
16. $x^2 + 2x > 3$
17. $x^2 + 2x - 3 < 0$
18. $x^2 - 4x - 1 > 0$
19. $x^2 + 8x - 5 \geq 0$
20. $-2x^2 + 6x + 15 \leq 0$
21. $x^3 - 3x^2 - x + 3 > 0$
22. $x^3 + 2x^2 - 4x - 8 \leq 0$
23. $x^3 - 2x^2 - 9x - 2 \geq -20$
24. $2x^3 + 13x^2 - 8x - 46 \geq 6$
25. $4x^2 - 4x + 1 \leq 0$
26. $x^2 + 3x + 8 > 0$

In Exercises 27–32, solve the inequality and write the solution set in interval notation.

27. $4x^3 - 6x^2 < 0$
28. $4x^3 - 12x^2 > 0$
29. $x^3 - 4x \geq 0$
30. $2x^3 - x^4 \leq 0$
31. $(x - 1)^2(x + 2)^3 \geq 0$
32. $x^4(x - 3) \leq 0$

**Graphical Analysis** In Exercises 33–36, use a graphing utility to graph the equation. Use the graph to approximate the values of $x$ that satisfy each inequality.

**Equation** | **Inequalities**
--- | ---
33. $y = -x^2 + 2x + 3$ | (a) $y \leq 0$ (b) $y \geq 3$
34. $y = \frac{1}{2}x^2 - 2x + 1$ | (a) $y \leq 0$ (b) $y \geq 7$
35. $y = \frac{1}{8}x^3 - \frac{1}{2}x$ | (a) $y \geq 0$ (b) $y \leq 6$
36. $y = x^3 - x^2 - 16x + 16$ | (a) $y \leq 0$ (b) $y \geq 36$

In Exercises 37–50, solve the inequality and graph the solution on the real number line.

37. $\frac{1}{x} - x > 0$
38. $\frac{1}{x} - 4 < 0$
39. $\frac{x + 6}{x + 1} - 2 < 0$
40. $\frac{x + 12}{x + 2} - 3 \geq 0$
41. $\frac{3x - 5}{x - 5} > 4$
42. $\frac{5 + 7x}{1 + 2x} < 4$
43. $\frac{4}{x + 5} > \frac{1}{2x + 3}$
44. $\frac{5}{x - 6} > \frac{3}{x + 2}$
45. $\frac{1}{x - 3} \leq \frac{9}{4x + 3}$
46. $\frac{1}{x} \geq \frac{1}{x + 3}$
47. $\frac{x^2 + 2x}{x^2 - 9} \leq 0$
48. $\frac{x^2 + 2x}{x^2 - 9} \leq 0$
49. $\frac{5}{x - 1} - \frac{2x}{x + 1} < 1$
50. $\frac{3x}{x - 1} \leq \frac{x}{x + 4} + 3$
Graphical Analysis  In Exercises 51–54, use a graphing utility to graph the equation. Use the graph to approximate the values of x that satisfy each inequality.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Inequalities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \frac{3x}{x - 2}$</td>
<td>(a) $y \leq 0$</td>
</tr>
<tr>
<td></td>
<td>(b) $y \geq 6$</td>
</tr>
<tr>
<td>$y = \frac{2(x - 2)}{x + 1}$</td>
<td>(a) $y \leq 0$</td>
</tr>
<tr>
<td></td>
<td>(b) $y \geq 8$</td>
</tr>
<tr>
<td>$y = \frac{2x^2}{x^2 + 4}$</td>
<td>(a) $y \geq 1$</td>
</tr>
<tr>
<td></td>
<td>(b) $y \leq 2$</td>
</tr>
<tr>
<td>$y = \frac{5x}{x^2 + 4}$</td>
<td>(a) $y \geq 1$</td>
</tr>
<tr>
<td></td>
<td>(b) $y \leq 0$</td>
</tr>
</tbody>
</table>

In Exercises 55–60, find the domain of x in the expression. Use a graphing utility to verify your result.

<table>
<thead>
<tr>
<th>x</th>
<th>55. $\sqrt{4 - x^2}$</th>
<th>56. $\sqrt{x^2 - 4}$</th>
<th>57. $\sqrt{x^2 - 7x + 12}$</th>
<th>58. $\sqrt{144 - 9x^2}$</th>
<th>59. $\sqrt{\frac{x}{x^2 - 2x - 35}}$</th>
<th>60. $\sqrt{\frac{x}{x^2 - 9}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In Exercises 61–66, solve the inequality. (Round your answers to two decimal places.)

61. $0.4x^2 + 5.26 < 10.2$
62. $-1.3x^2 + 3.78 > 2.12$
63. $-0.5x^2 + 12.5x + 1.6 > 0$
64. $1.2x^2 + 4.8x + 3.1 < 5.3$
65. $\frac{1}{2.3x - 5.2} > 3.4$
66. $\frac{2}{3.1x - 3.7} > 5.8$

67. Height of a Projectile  A projectile is fired straight upward from ground level with an initial velocity of 160 feet per second.
(a) At what instant will it be back at ground level?
(b) When will the height exceed 384 feet?

68. Height of a Projectile  A projectile is fired straight upward from ground level with an initial velocity of 128 feet per second.
(a) At what instant will it be back at ground level?
(b) When will the height be less than 128 feet?

69. Geometry  A rectangular playing field with a perimeter of 100 meters is to have an area of at least 500 square meters. Within what bounds must the length of the rectangle lie?

70. Geometry  A rectangular parking lot with a perimeter of 440 feet is to have an area of at least 8000 square feet. Within what bounds must the length of the rectangle lie?

71. Cost, Revenue, and Profit  The revenue and cost equations for a product are
$$R = x(75 - 0.0005x) \quad \text{and} \quad C = 30x + 250,000$$
where $R$ and $C$ are measured in dollars and $x$ represents the number of units sold. How many units must be sold to obtain a profit of at least $750,000? What is the price per unit?

72. Cost, Revenue, and Profit  The revenue and cost equations for a product are
$$R = x(50 - 0.0002x) \quad \text{and} \quad C = 12x + 150,000$$
where $R$ and $C$ are measured in dollars and $x$ represents the number of units sold. How many units must be sold to obtain a profit of at least $1,650,000? What is the price per unit?

73. Cable Television  The percents $C$ of households in the United States that owned a television and had cable from 1980 to 2003 can be modeled by
$$C = 0.0031t^3 - 0.216t^2 + 5.54t + 19.1, \quad 0 \leq t \leq 23$$
where $t$ is the year, with $t = 0$ corresponding to 1980.
(Source: Nielsen Media Research)
(a) Use a graphing utility to graph the equation.
(b) Complete the table to determine the year in which the percent of households that own a television and have cable will exceed 75%.

\[
\begin{array}{cccccccc}
\hline
\text{t} & 24 & 26 & 28 & 30 & 32 & 34 \\
\text{C} & & & & & & \\
\hline
\end{array}
\]
(c) Use the trace feature of a graphing utility to verify your answer to part (b).
(d) Complete the table to determine the years during which the percent of households that own a television and have cable will be between 85% and 100%.

\[
\begin{array}{ccccccccc}
\hline
\text{t} & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 \\
\text{C} & & & & & & & & \\
\hline
\end{array}
\]
(e) Use the trace feature of a graphing utility to verify your answer to part (d).
(f) Explain why the model may give values greater than 100% even though such values are not reasonable.
74. **Safe Load** The maximum safe load uniformly distributed over a one-foot section of a two-inch-wide wooden beam is approximated by the model $\text{Load} = 168.5d^2 - 472.1$, where $d$ is the depth of the beam.

(a) Evaluate the model for $d = 4, d = 6, d = 8, d = 10$, and $d = 12$. Use the results to create a bar graph.

(b) Determine the minimum depth of the beam that will safely support a load of 2000 pounds.

75. **Resistors** When two resistors of resistances $R_1$ and $R_2$ are connected in parallel (see figure), the total resistance $R$ satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$ 

Find $R_1$ for a parallel circuit in which $R_2 = 2$ ohms and $R$ must be at least 1 ohm.

76. **Education** The numbers $N$ (in thousands) of master’s degrees earned by women in the United States from 1990 to 2002 are approximated by the model

$$N = -0.03t^2 + 9.6t + 172$$

where $t$ represents the year, with $t = 0$ corresponding to 1990 (see figure). (Source: U.S. National Center for Education Statistics)

![Graph of master's degrees earned by women](image)

(a) According to the model, during what year did the number of master’s degrees earned by women exceed 220,000?

(b) Use the graph to verify the result of part (a).

(c) According to the model, during what year will the number of master’s degrees earned by women exceed 320,000?

(d) Use the graph to verify the result of part (c).

---

**Synthesis**

**True or False?** In Exercises 77 and 78, determine whether the statement is true or false. Justify your answer.

77. The zeros of the polynomial $x^3 - 2x^2 - 11x + 12 \geq 0$ divide the real number line into four test intervals.

78. The solution set of the inequality $\frac{1}{2}x^2 + 3x + 6 \geq 0$ is the entire set of real numbers.

**Exploration** In Exercises 79–82, find the interval for $b$ such that the equation has at least one real solution.

79. $x^2 + bx + 4 = 0$

80. $x^2 + bx - 4 = 0$

81. $3x^2 + bx + 10 = 0$

82. $2x^2 + bx + 5 = 0$

83. (a) Write a conjecture about the intervals for $b$ in Exercises 79–82. Explain your reasoning.

(b) What is the center of each interval for $b$ in Exercises 79–82?

84. Consider the polynomial $(x - a)(x - b)$ and the real number line shown below.

![Diagram of real number line](image)

(a) Identify the points on the line at which the polynomial is zero.

(b) In each of the three subintervals of the line, write the sign of each factor and the sign of the product.

(c) For what $x$-values does the polynomial change signs?

---

**Skills Review**

In Exercises 85–88, factor the expression completely.

85. $4x^2 + 20x + 25$

86. $(x + 3)^2 - 16$

87. $x^2(x + 3) - 4(x + 3)$

88. $2x^4 - 54x$

In Exercises 89 and 90, write an expression for the area of the region.

89.

![Diagram of region 89](image)

90.

![Diagram of region 90](image)
## Chapter Summary

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</table>
1.1 In Exercises 1–4, complete a table of values. Use the solution points to sketch the graph of the equation.

1. \( y = 3x - 5 \)  
2. \( y = -\frac{1}{2}x + 2 \)  
3. \( y = x^2 - 3x \)  
4. \( y = 2x^2 - x - 9 \)

In Exercises 5–10, sketch the graph by hand.

5. \( y - 2x - 3 = 0 \)  
6. \( 3x + 2y + 6 = 0 \)  
7. \( y = \sqrt{5 - x} \)  
8. \( y = \sqrt{x + 2} \)  
9. \( y + 2x^2 = 0 \)  
10. \( y = x^2 - 4x \)

In Exercises 11 and 12, find the \( x \)- and \( y \)-intercepts of the graph of the equation.

11. \( y = (x - 3)^2 - 4 \)  
12. \( y = |x + 1| - 3 \)

In Exercises 13–20, use the algebraic tests to check for symmetry with respect to both axes and the origin. Then sketch the graph of the equation.

13. \( y = -4x + 1 \)  
14. \( y = 5x - 6 \)  
15. \( y = 5 - x^2 \)  
16. \( y = x^2 - 10 \)  
17. \( y = x^3 + 3 \)  
18. \( y = -6 - x^3 \)  
19. \( y = \sqrt{x + 5} \)  
20. \( y = |x| + 9 \)

In Exercises 21–26, find the center and radius of the circle and sketch its graph.

21. \( x^2 + y^2 = 9 \)  
22. \( x^2 + y^2 = 4 \)  
23. \( (x + 2)^2 + y^2 = 16 \)  
24. \( x^2 + (y - 8)^2 = 81 \)  
25. \( (x - \frac{1}{2})^2 + (y + 1)^2 = 36 \)  
26. \( (x + 4)^2 + (y - \frac{3}{2})^2 = 100 \)  
27. Find the standard form of the equation of the circle for which the endpoints of a diameter are \((0, 0)\) and \((4, -6)\).

28. Find the standard form of the equation of the circle for which the endpoints of a diameter are \((-2, -3)\) and \((4, -10)\).

29. **Physics** The force \( F \) (in pounds) required to stretch a spring \( x \) inches from its natural length (see figure) is

\[
F = \frac{5}{4}x, \quad 0 \leq x \leq 20.
\]

(a) Use the model to complete the table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force, ( F )</td>
<td>( \frac{5}{4} )</td>
<td>( \frac{10}{4} )</td>
<td>( \frac{20}{4} )</td>
<td>( \frac{24}{4} )</td>
<td>( \frac{30}{4} )</td>
<td>( \frac{30}{4} )</td>
</tr>
</tbody>
</table>

(b) Sketch a graph of the model.

(c) Use the graph to estimate the force necessary to stretch the spring 10 inches.

30. **Number of Stores** The numbers \( N \) of Target stores for the years 1994 to 2003 can be approximated by the model

\[
N = 3.69t^2 + 939, \quad 4 \leq t \leq 13
\]

where \( t \) is the time (in years), with \( t = 4 \) corresponding to 1994. (Source: Target Corp.)

(a) Sketch a graph of the model.

(b) Use the graph to estimate the year in which the number of stores was 1300.

1.2 In Exercises 31–34, determine whether the equation is an identity or a conditional equation.

31. \( 6 - (x - 2)^2 = 2 + 4x - x^2 \)
32. \( 3(x - 2) + 2x = 2(x + 3) \)
33. \( -x^3 + x(7 - x) + 3 = x(-x^2 - x) + 7(x + 1) - 4 \)
34. \( 3(x^2 - 4x + 8) = -10(x + 2) - 3x^2 + 6 \)
In Exercises 35–42, solve the equation (if possible) and check your solution.

35. \(3x - 2(x + 5) = 10\)
36. \(4x + 2(7 - x) = 5\)
37. \(4(x + 3) - 3 = 2(4 - 3x) - 4\)
38. \(\frac{1}{2}(x - 3) - 2(x + 1) = 5\)
39. \(\frac{x}{5} - 3 = \frac{x}{3} + 1\)
40. \(\frac{4x - 3}{6} + \frac{x}{4} = x - 2\)
41. \(\frac{18}{x} = \frac{10}{x - 4}\)
42. \(\frac{5}{x - 2} = \frac{13}{2x - 3}\)

In Exercises 43–50, find the \(x\)- and \(y\)-intercepts of the graph of the equation algebraically.

43. \(y = 3x - 1\)
44. \(y = -5x + 6\)
45. \(y = 2(x - 4)\)
46. \(y = 4(7x + 1)\)
47. \(y = -\frac{1}{3}x + \frac{5}{3}\)
48. \(y = \frac{3}{2}x - \frac{1}{4}\)
49. \(3.8y - 0.5x + 1 = 0\)
50. \(1.5y + 2x - 1.2 = 0\)

51. Geometry The surface area \(S\) of the cylinder shown in the figure is approximated by

\[ S = 2(3.14)(3)^2 + 2(3.14)(3)h. \]

The surface area is 244.92 square inches. Find the height \(h\) of the cylinder.

52. Temperature The Fahrenheit and Celsius temperature scales are related by the equation

\[ C = \frac{5}{9}F - \frac{160}{9}. \]

Find the Fahrenheit temperature that corresponds to 100°C Celsius.

In October, a greeting card company’s total profit was 12% more than it was in September. The total profit for the two months was $689,000. Write a verbal model, assign labels, and write an algebraic equation to find the profit for each month.

54. Discount The price of a digital camera has been discounted $85. The sale price is $340. Write a verbal model, assign labels, and write an algebraic equation to find the percent discount.

55. Shadow Length A person who is 6 feet tall walks away from a streetlight toward the tip of the streetlight’s shadow. When the person is 15 feet from the streetlight, the tip of the person’s shadow and the shadow cast by the streetlight coincide at a point 5 feet in front of the person (see figure). How tall is the streetlight?

56. Finance A group agrees to share equally in the cost of a $48,000 piece of machinery. If it can find two more group members, each member’s share will decrease by $4000. How many are presently in the group?

57. Business Venture You are planning to start a small business that will require an investment of $90,000. You have found some people who are willing to share equally in the venture. If you can find three more people, each person’s share will decrease by $2500. How many people have you found so far?

58. Average Speed You commute 56 miles one way to work. The trip to work takes 10 minutes longer than the trip home. Your average speed on the trip home is 8 miles per hour faster. What is your average speed on the trip home?

59. Mixture Problem A car radiator contains 10 liters of a 30% antifreeze solution. How many liters will have to be replaced with pure antifreeze if the resulting solution is to be 50% antifreeze?

60. Investment You invested $6000 at 4\(\frac{1}{2}\)% and 5\(\frac{1}{2}\)% simple interest. During the first year, the two accounts earned $305. How much did you invest in each fund? (Note: The 5\(\frac{1}{2}\)% account is more risky.)
In Exercises 61 and 62, solve for the indicated variable.

61. **Volume of a Cone**
   Solve for \( h \): \( V = \frac{1}{3} \pi r^2 h \)

62. **Kinetic Energy**
   Solve for \( m \): \( E = \frac{1}{2} m v^2 \)

63. **Travel Time** Two cars start at a given time and travel in the same direction at average speeds of 40 miles per hour and 55 miles per hour. How much time will elapse before the two cars are 10 miles apart?

64. **Geometry** The volume of a circular cylinder is \( 81 \pi \) cubic feet. The cylinder’s radius is 3 feet. What is the height of the cylinder?

1.4 In Exercises 65–74, use any method to solve the quadratic equation.

65. \( 15 + x - 2x^2 = 0 \)
66. \( 2x^2 - x - 28 = 0 \)
67. \( 6 = 3x^2 \)
68. \( 16x^2 = 25 \)
69. \( (x + 4)^2 = 18 \)
70. \( (x - 8)^2 = 15 \)
71. \( x^2 - 12x + 30 = 0 \)
72. \( x^2 + 6x - 3 = 0 \)
73. \( -2x^2 - 5x + 27 = 0 \)
74. \( -20 - 3x + 3x^2 = 0 \)

75. **Simply Supported Beam** A simply supported 20-foot beam supports a uniformly distributed load of 1000 pounds per foot. The bending moment \( M \) (in foot-pounds) \( x \) feet from one end of the beam is given by \( M = 500x(20 - x) \).

   (a) Where is the bending moment zero?
   (b) Use a graphing utility to graph the equation.
   (c) Use the graph to determine the point on the beam where the bending moment is the greatest.

76. **Sports** You throw a softball straight up into the air at a velocity of 30 feet per second. You release the softball at a height of 5.8 feet and catch it when it falls back to a height of 6.2 feet.

   (a) Use the position equation to write a mathematical model for the height of the softball.
   (b) What is the height of the softball after 1 second?
   (c) How many seconds is the softball in the air?

1.5 In Exercises 77–80, write the complex number in standard form.

77. \( 6 + \sqrt{-4} \)
78. \( 3 - \sqrt{-25} \)
79. \( i^2 + 3i \)
80. \(-5i + i^2 \)

In Exercises 81–86, perform the operation and write the result in standard form.

81. \( (7 + 5i) + (-4 + 2i) \)
82. \( \left( \frac{\sqrt{5}}{2} - \frac{\sqrt{2}}{2} i \right) - \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \right) \)
83. \( 5i(13 - 8i) \)
84. \( (1 + 6i)(5 - 2i) \)
85. \( (10 - 8i)(2 - 3i) \)
86. \( i(6 + i)(3 - 2i) \)

In Exercises 87 and 88, write the quotient in standard form.

87. \( \frac{6 + i}{4 - i} \)
88. \( \frac{3 + 2i}{5 + i} \)

In Exercises 89 and 90, perform the operation and write the result in standard form.

89. \( \frac{4}{2 - 3i} + \frac{2}{1 + i} \)
90. \( \frac{1}{2 + i} - \frac{5}{1 + 4i} \)

In Exercises 91–94, find all solutions of the equation.

91. \( 3x^2 + 1 = 0 \)
92. \( 2 + 8x^2 = 0 \)
93. \( x^2 - 2x + 10 = 0 \)
94. \( 6x^2 + 3x + 27 = 0 \)

1.6 In Exercises 95–114, find all solutions of the equation. Check your solutions in the original equation.

95. \( 5x^4 - 12x^3 = 0 \)
96. \( 4x^3 - 6x^2 = 0 \)
97. \( x^4 - 5x^2 + 6 = 0 \)
98. \( 9x^4 + 27x^3 - 4x^2 - 12x = 0 \)
99. \( \sqrt{x + 4} = 3 \)
100. \( \sqrt{x - 2} = 8 \)
101. \( \sqrt{2x + 3} + \sqrt{x - 2} = 2 \)
102. \( 5\sqrt{x} - \sqrt{x - 1} = 6 \)
103. \((x - 1)^{2/3} - 25 = 0 \)
104. \((x + 2)^{3/4} = 27 \)
105. \((x + 4)^{1/2} + 5x(x + 4)^{3/2} = 0 \)
106. \(8x^2(x^2 - 4)^{1/3} + (x^2 - 4)^{4/3} = 0 \)
107. \( \frac{5}{x} = 1 + \frac{3}{x + 2} \)
108. \( \frac{6}{x} + \frac{8}{x + 5} = 3 \)
109. \( \frac{3}{x + 2} - \frac{1}{x} = \frac{1}{5x} \)
110. \( \frac{12}{x + 5} + \frac{5}{x} = \frac{20}{x} \)
111. \( |x - 5| = 10 \)
112. \( |2x + 3| = 7 \)
113. \( |x^2 - 3| = 2x \)
114. \( |x^2 - 6| = x \)

115. **Demand** The demand equation for a hair dryer is

\[ p = 42 - \sqrt{0.001x + 2} \]

where \( x \) is the number of units demanded per day and \( p \) is the price per unit. Find the demand if the price is set at $29.95.
116. **Data Analysis: Newspapers**  The total numbers \( N \) of daily evening newspapers in the United States from 1970 to 2000 can be approximated by the model
\[
N = 1481 - 4.6t^{3/2}, \quad 0 \leq t \leq 30
\]
where \( t \) represents the year, with \( t = 0 \) corresponding to 1970. The actual numbers of newspapers for selected years are shown in the table.  (Source: Editor & Publisher Co.)

<table>
<thead>
<tr>
<th>Year</th>
<th>Newspapers, ( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1970</td>
<td>1429</td>
</tr>
<tr>
<td>1975</td>
<td>1436</td>
</tr>
<tr>
<td>1980</td>
<td>1388</td>
</tr>
<tr>
<td>1985</td>
<td>1220</td>
</tr>
<tr>
<td>1990</td>
<td>1084</td>
</tr>
<tr>
<td>1995</td>
<td>981</td>
</tr>
<tr>
<td>2000</td>
<td>727</td>
</tr>
</tbody>
</table>

(a) Use a graphing utility to plot the data and graph the model in the same viewing window. How well does the model fit the data?

(b) Use the graph in part (a) to estimate the year in which there were 800 daily evening newspapers.

(c) Use the model to verify algebraically the estimate from part (b).

132. **Cost, Revenue, and Profit**  The revenue for selling \( x \) units of a product is \( R = 125.33x \). The cost of producing \( x \) units is \( C = 92x + 1200 \). To obtain a profit, the revenue must be greater than the cost. Determine the smallest value of \( x \) for which this product returns a profit.

1.8  In Exercises 133–144, solve the inequality.

133. \( x^2 - 6x - 27 < 0 \)
134. \( x^2 - 2x \geq 3 \)
135. \( 6x^2 + 5x < 4 \)
136. \( 2x^2 + x \geq 15 \)
137. \( x^3 - 16x \geq 0 \)
138. \( 12x^3 - 20x^2 < 0 \)
139. \( \frac{x + 8}{x + 5} - 2 < 0 \)
140. \( \frac{3x + 8}{x - 3} \leq 4 \)
141. \( \frac{2}{x + 1} \leq \frac{3}{x - 1} \)
142. \( \frac{x - 5}{3 - x} < 0 \)
143. \( \frac{x^2 + 7x + 12}{x} \geq 0 \)
144. \( \frac{1}{x - 2} > \frac{1}{x} \)

In Exercises 137–140, write an inequality that represents the interval and state whether the interval is bounded or unbounded.

117. \((-7, 2]\)
118. \((4, \infty)\)
119. \((-\infty, -10]\)
120. \([-2, 2]\)

In Exercises 121–130, solve the inequality.

121. \(9x - 8 \leq 7x + 16\)
122. \(\frac{15}{7}x + 4 > 3x - 5\)
123. \(4(5 - 2x) \leq \frac{1}{2}(8 - x)\)
124. \(\frac{1}{3}(3 - x) > \frac{1}{2}(2 - 3x)\)
125. \(-19 < 3x - 17 \leq 34\)
126. \(-3 \leq \frac{2x - 5}{3} < 5\)
127. \(|x| \leq 4\)
128. \(|x - 2| < 1\)
129. \(|x - 3| > 4\)
130. \(|x - \frac{3}{2}| \geq \frac{3}{2}\)

131. **Geometry**  The side of a square is measured as \(19.3\) centimeters with a possible error of \(0.5\) centimeter. Using these measurements, determine the interval containing the area of the square.

145. **Investment**  \(P\) dollars invested at interest rate \(r\) compounded annually increases to an amount
\[
A = P(1 + r)^2
\]
in 2 years. An investment of \$5000 is to increase to an amount greater than \$5500 in 2 years. The interest rate must be greater than what percent?

146. **Population of a Species**  A biologist introduces 200 ladybugs into a crop field. The population \(P\) of the ladybugs is approximated by the model
\[
P = \frac{1000(1 + 3t)}{5 + t}
\]
where \(t\) is the time in days. Find the time required for the population to increase to at least 2000 ladybugs.

**Synthesis**

**True or False?**  In Exercises 147 and 148, determine whether the statement is true or false. Justify your answer.

147. \(\sqrt{-18} \sqrt{-2} = \sqrt{(-18)(-2)}\)
148. The equation \(325x^2 - 717x + 398 = 0\) has no solution.

149. **Writing**  Explain why it is essential to check your solutions to radical, absolute value, and rational equations.

150. **Error Analysis**  What is wrong with the following solution?

\[
|11x + 4| \geq 26 \quad \text{or} \quad 11x + 4 \geq 26
\]

\[
11x \leq 22 \quad \text{and} \quad 11x \geq 22
\]

\[
x \geq 2
\]

\[
x \leq 2
\]
Chapter Test

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1–6, check for symmetry with respect to both axes and the origin. Then sketch the graph of the equation. Identify any \( x \)- and \( y \)-intercepts.

1. \( y = 4 - \frac{3}{4}x \)
2. \( y = 4 - \frac{3}{4}|x| \)
3. \( y = 4 - (x - 2)^2 \)
4. \( y = x - x^3 \)
5. \( y = \sqrt{3} - x \)
6. \( (x - 3)^2 + y^2 = 9 \)

In Exercises 7–12, solve the equation (if possible).

7. \( \frac{2}{3}(x - 1) + \frac{1}{2}x = 10 \)
8. \( (x - 3)(x + 2) = 14 \)
9. \( \frac{x - 2}{x + 2} + \frac{4}{x + 2} + 4 = 0 \)
10. \( x^4 + x^2 - 6 = 0 \)
11. \( 2\sqrt{x} - \sqrt{2x + 1} = 1 \)
12. \( |3x - 1| = 7 \)

In Exercises 13–16, solve the inequality. Sketch the solution set on the real number line.

13. \( -3 \leq 2(x + 4) < 14 \)
14. \( \frac{2}{x} > \frac{5}{x + 6} \)
15. \( 2x^2 + 5x > 12 \)
16. \( |x - 15| \geq 5 \)

17. Perform each operation and write the result in standard form.
   (a) \( 10i - (3 + \sqrt{-25}) \)
   (b) \( (2 + \sqrt{3}i)(2 - \sqrt{3}i) \)

18. Write the quotient in standard form: \( \frac{5}{2 + i} \)

19. The sales \( y \) (in billions of dollars) for Dell, Inc. from 1994 to 2003 can be approximated by the model
   \( y = 4.45t - 16.6, \quad 4 \leq t \leq 13 \)
   where \( t \) is the time (in years), with \( t = 4 \) corresponding to 1994. (Source: Dell, Inc.)
   (a) Sketch a graph of the model.
   (b) Assuming that the pattern continues, use the graph in part (a) to estimate the sales in 2008.
   (c) Use the model to verify algebraically the estimate from part (b).

20. A basketball has a volume of about 455.9 cubic inches. Find the radius of the basketball (accurate to three decimal places).

21. On the first part of a 350-kilometer trip, a salesperson travels 2 hours and 15 minutes at an average speed of 100 kilometers per hour. The salesperson needs to arrive at the destination in another hour and 20 minutes. Find the average speed required for the remainder of the trip.

22. The area of the ellipse in the figure at the left is \( A = \pi ab \). If \( a \) and \( b \) satisfy the constraint \( a + b = 100 \), find \( a \) and \( b \) such that the area of the ellipse equals the area of the circle.
Conditional Statements

Many theorems are written in the if-then form “if $p$, then $q$,” which is denoted by $p \rightarrow q$. Here are some other ways to express the conditional statement $p \rightarrow q$.

- $p$ implies $q$.  
- $p$, only if $q$.  
- $p$ is sufficient for $q$.

Conditional statements can be either true or false. The conditional statement $p \rightarrow q$ is false only when $p$ is true and $q$ is false. To show that a conditional statement is true, you must prove that the conclusion follows for all cases that fulfill the hypothesis. To show that a conditional statement is false, you need only to describe a single counterexample that shows that the statement is not always true.

For instance, $x = -4$ is a counterexample that shows that the following statement is false.

If $x^2 = 16$, then $x = 4$.

The hypothesis “$x^2 = 16$” is true because $(-4)^2 = 16$. However, the conclusion “$x = 4$” is false. This implies that the given conditional statement is false.

For the conditional statement $p \rightarrow q$, there are three important associated conditional statements.

1. The converse of $p \rightarrow q$: $q \rightarrow p$
2. The inverse of $p \rightarrow q$: $\neg p \rightarrow \neg q$
3. The contrapositive of $p \rightarrow q$: $\neg q \rightarrow \neg p$

The symbol $\neg$ means the negation of a statement. For instance, the negation of “The engine is running” is “The engine is not running.”

Example: Writing the Converse, Inverse, and Contrapositive

Write the converse, inverse, and contrapositive of the conditional statement “If I get a B on my test, then I will pass the course.”

Solution

- Converse: If I pass the course, then I got a B on my test.
- Inverse: If I do not get a B on my test, then I will not pass the course.
- Contrapositive: If I do not pass the course, then I did not get a B on my test.

In the example above, notice that neither the converse nor the inverse is logically equivalent to the original conditional statement. On the other hand, the contrapositive is logically equivalent to the original conditional statement.
1. Let \( x \) represent the time (in seconds) and let \( y \) represent the distance (in feet) between you and a tree. Sketch a possible graph that shows how \( x \) and \( y \) are related if you are walking toward the tree.

2. (a) Find the following sums
   \[
   1 + 2 + 3 + 4 + 5 = \_
   \]
   \[
   1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \_
   \]
   \[
   1 + 2 + 3 + 4 + 5 + 6
   + 7 + 8 + 9 + 10 = \_
   \]
   (b) Use the following formula for the sum of the first \( n \) natural numbers to verify your answers to part (a).
   \[
   1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n + 1)
   \]
   (c) Use the formula in part (b) to find \( n \) if the sum of the first \( n \) natural numbers is 210.

3. The area of an ellipse is given by \( A = \pi ab \) (see figure). For a certain ellipse, it is required that \( a + b = 20 \).

   (a) Show that \( A = \pi a(20 - a) \).
   (b) Complete the table.

<table>
<thead>
<tr>
<th>( a )</th>
<th>4</th>
<th>7</th>
<th>10</th>
<th>13</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   (c) Find two values of \( a \) such that \( A = 300 \).
   (d) Use a graphing utility to graph the area equation.
   (e) Find the \( x \)-intercepts of the graph of the area equation. What do these values represent?
   (f) What is the maximum area? What values of \( a \) and \( b \) yield the maximum area?

4. A building code requires that a building be able to withstand a certain amount of wind pressure. The pressure \( P \) (in pounds per square foot) from wind blowing at \( s \) miles per hour is given by
   \[
   P = 0.00256 s^2.
   \]
   (a) A two-story library is designed. Buildings this tall are often required to withstand wind pressure of 20 pounds per square foot. Under this requirement, how fast can the wind be blowing before it produces excessive stress on the building?
   (b) To be safe, the library is designed so that it can withstand wind pressure of 40 pounds per square foot. Does this mean that the library can survive wind blowing at twice the speed you found in part (a)? Justify your answer.
   (c) Use the pressure formula to explain why even a relatively small increase in the wind speed could have potentially serious effects on a building.

5. For a bathtub with a rectangular base, Toricelli’s Law implies that the height \( h \) of water in the tub \( t \) seconds after it begins draining is given by
   \[
   h = \left( h_0 - \frac{2\pi d^2}{lw} \sqrt{3} t \right)^2
   \]
   where \( l \) and \( w \) are the tub’s length and width, \( d \) is the diameter of the drain, and \( h_0 \) is the water’s initial height. (All measurements are in inches.) You completely fill a tub with water. The tub is 60 inches long by 30 inches wide by 25 inches high and has a drain with a two-inch diameter.
   (a) Find the time it takes for the tub to go from being full to half-full.
   (b) Find the time it takes for the tub to go from being half-full to empty.
   (c) Based on your results in parts (a) and (b), what general statement can you make about the speed at which the water drains?

6. (a) Consider the sum of squares \( x^2 + 9 \). If the sum can be factored, then there are integers \( m \) and \( n \) such that
   \[
   x^2 + 9 = (x + m)(x + n).
   \]
   Write two equations relating the sum and the product of \( m \) and \( n \) to the coefficients in \( x^2 + 9 \).
   (b) Show that there are no integers \( m \) and \( n \) that satisfy both equations you wrote in part (a). What can you conclude?
7. A Pythagorean Triple is a group of three integers, such as 3, 4, and 5, that could be the lengths of the sides of a right triangle.

(a) Find two other Pythagorean Triples.
(b) Notice that 3 · 4 · 5 = 60. Is the product of the three numbers in each Pythagorean Triple evenly divisible by 3? by 4? by 5?
(c) Write a conjecture involving Pythagorean Triples and divisibility by 60.

8. Determine the solutions \( x_1 \) and \( x_2 \) of each quadratic equation. Use the values of \( x_1 \) and \( x_2 \) to fill in the boxes.

<table>
<thead>
<tr>
<th>Equation</th>
<th>( x_1 ), ( x_2 )</th>
<th>( x_1 + x_2 )</th>
<th>( x_1 \cdot x_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( x^2 - x - 6 = 0 )</td>
<td>[ ]</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
<tr>
<td>(b) ( 2x^2 + 5x - 3 = 0 )</td>
<td>[ ]</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
<tr>
<td>(c) ( 4x^2 - 9 = 0 )</td>
<td>[ ]</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
<tr>
<td>(d) ( x^2 - 10x + 34 = 0 )</td>
<td>[ ]</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
</tbody>
</table>

9. Consider a general quadratic equation

\[ ax^2 + bx + c = 0 \]

whose solutions are \( x_1 \) and \( x_2 \). Use the results of Exercise 8 to determine a relationship among the coefficients \( a, b, \) and \( c \) and the sum \( x_1 + x_2 \) and the product \( x_1 \cdot x_2 \) of the solutions.

10. (a) The principal cube root of \( 125, \sqrt[3]{125} \), is 5. Evaluate the expression \( x^3 \) for each value of \( x \).

(i) \( x = \frac{-5 + 5\sqrt{3}i}{2} \)

(ii) \( x = \frac{-5 - 5\sqrt{3}i}{2} \)

(b) The principal cube root of \( 27, \sqrt[3]{27} \), is 3. Evaluate the expression \( x^3 \) for each value of \( x \).

(i) \( x = \frac{-3 + 3\sqrt{3}i}{2} \)

(ii) \( x = \frac{-3 - 3\sqrt{3}i}{2} \)

(c) Use the results of parts (a) and (b) to list possible cube roots of (i), 1, (ii) 8, and (iii) 64. Verify your results algebraically.

11. The multiplicative inverse of \( z \) is a complex number \( z_m \) such that \( z \cdot z_m = 1 \). Find the multiplicative inverse of each complex number.

(a) \( z = 1 + i \)  \quad (b) \( z = 3 - i \)  \quad (c) \( z = -2 + 8i \)

12. Prove that the product of a complex number \( a + bi \) and its complex conjugate is a real number.

13. A fractal is a geometric figure that consists of a pattern that is repeated infinitely on a smaller and smaller scale. The most famous fractal is called the Mandelbrot Set, named after the Polish-born mathematician Benoit Mandelbrot. To draw the Mandelbrot Set, consider the following sequence of numbers.

\[ c, c^2 + c, (c^2 + c)^2 + c, [(c^2 + c)^2 + c]^2 + c, \ldots \]

The behavior of this sequence depends on the value of the complex number \( c \). If the sequence is bounded (the absolute value of each number in the sequence, \(|a + bi| = \sqrt{a^2 + b^2}\), is less than some fixed number \( N \)), the complex number \( c \) is in the Mandelbrot Set, and if the sequence is unbounded (the absolute value of the terms of the sequence become infinitely large), the complex number \( c \) is not in the Mandelbrot Set. Determine whether the complex number \( c \) is in the Mandelbrot Set.

(a) \( c = i \)  \quad (b) \( c = 1 + i \)  \quad (c) \( c = -2 \)

The figure below shows a black and yellow photo of the Mandelbrot Set.

14. Use the equation \( 4\sqrt{x} = 2x + k \) to find three different values of \( k \) such that the equation has two solutions, one solution, and no solution. Describe the process you used to find the values.

15. Use the graph of \( y = x^4 - 3x^2 - 4x + 8 \) to solve the inequality \( x^4 - 3x^2 - 4x + 8 > 0 \).

16. When you buy a 16-ounce bag of chips, you expect to get precisely 16 ounces. The actual weight \( w \) (in ounces) of a “16-ounce” bag of chips is given by

\[ |w - 16| \leq \frac{1}{2}. \]

You buy four 16-ounce bags. What is the greatest amount you can expect to get? What is the smallest amount? Explain.