Developing and Composing With Scales based on Recurrent Sequences

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Abstract:

Following up on suggestions by Ervin Wilson, who showed that an infinity of recurrent sequences can be obtained from taking the sums of the diagonals of Pingala s Meru Prastala, known in the West as Pascal s Triangle, the author has generated a number of scales using these sequences. Among the sequences used are familiar ones such as the Fibonacci series, and other more exotic sequences. Just intonation scales made directly from the proportions of the number series are made, as well as Pythagorean scales made from taking the Means the series tend to, and treating them as generating intervals for scales. The resulting scales are then divided into pseudo -White Key and -Black Key subsets using the Moments of Symmetry technique of Wilson and Chalmers. Examples of these scales, and the harmonies implied by them are heard as musical examples. Compositional work with these scales is then discussed, including using the intervals of the scales not only as generators for musical harmony, but also as generators for rhythm, timbre and spatialisation. A discussion of the spatial nature of a sound as an emergent property of its harmonic content concludes the paper

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For some time now, Ervin Wilson has been suggesting that I would be interested in investigating the properties of scales made with recurrent sequences. His 1993 article, The Scales of Mt. Meru (Wilson, 1993a), following up on work by Thomas M. Green (Green, 1968) and A. N. Singh (Singh, 1936), shows how an infinite series of recurrent sequences can be derived by taking the sums of different diagonals of Pingala s Meru Prastara, otherwise known as Pascal s triangle. This diagram was first described by the Hindu mathematician Pingala as early as 200 BCE, and other triangular number diagrams existed in Hindu mathematics as early as 2000 BCE, according to Ernest McClain. (McClain, 1976). Ervin Wilson has suggested that the recurrent sequences found in the Meru Prastara could be useful as a source for musical scales.

A recurrent sequence is any sequence of numbers generated by following a rule for adding previous elements of the sequence to get the next one. For example, the fibonacci series, one of the most famous of these, follows the formula An = An-2 + An-1. That is, each new number An is the sum of the two previous elements in the series, the number just before it (An-1) and the number two elements before it (An-2). This yields the famous sequence 1 1 2 3 5 8 13 21 34 55 89 etc. In his 1993 article, Wilson enumerated the first 192 recurrent sequences that could be yielded by summing the elements found on different diagonals on Pascals triangle. A few examples will suffice - the first five sequences are:

An = An-2 + An-1 (An/An-1 converges on 1.618033989...)
Bn = Bn-3 + Bn-1 (Bn/Bn-1 converges on 1.465571232...)
Cn = Cn-3 + Cn-2 (Cn/Cn-1 converges on 1.324717957...)
Dn = Dn-4 + Dn-1 (Dn/Dn-1 converges on 1.380277569...)
En = En-4 + En-3 (En/En-1 converges on 1.220744085...)

In each of these sequences, the farther one goes out them, the more the ratio between any two elements comes closer to a mean. The means that each of the above five series approach are given in the parentheses after the formula for that series.

While each of these series is interesting in its own right, the problem of how to turn them into scales puzzled me for some time. At first I thought I might just take the interval between each successive two elements of the series and treat it as a pitch in a just intonation scale. So, for example, for the fibonacci series, that would mean a scale consisting of the ratios 2/1, 3/2, 5/3, 8/5, 13/8, etc. Putting these ratios and their inversions (4/3, 6/5, 5/4, 16/13, etc.) into Emanuel Op de Coul s Scala program yielded the following scale, which though it might be charming, was not too useful. The preponderance of intervals around 833 cents and 366 cents can be explained by the fact that as you go further and further out the series, the ratio between each elements tends towards 1.618etc....which, expressed as the ratio of two musical frequencies, is an interval of around 833 cents. The inversion of this interval gives the interval of around 366 cents. So, this clearly wasn t the way to go.

Scale made by taking each fibonacci pair as a ratio expressing a scale degree.

0:	1/1	0.000
1:	6/5	315.641
2:	16/13	359.472
3:	21/17	365.825
4:	110/89	366.752
5:	288/233	366.887
6:	89/72	366.970
7:	68/55	367.324
8:	26/21	369.747
9:	5/4	386.314
10:	4/3	498.045
11:	3/2	701.955
12:	8/5	813.686
13:	21/13	830.253
14:	55/34	832.676
15:	144/89	833.030
16:	233/144	833.113
17:	89/55	833.248
18:	34/21	834.175
19:	13/8	840.528
20:	5/3	884.359
21:	2/1	1200.000

Then I thought, What about using the ratios of successive integers as intervals and pile them up, like a chain of 3/2s, but here each element in the chain is different? So here you would have 3/2 + 5/3 + 8/5 + 13/21 etc. as intervals and then you could reduce your chain of intervals into an octave. I did this with the first 12 elements of the fibonacci series, and here is what I got:

Degree	Ratio	Cents	Element of series
0:	1/1	0.000	(1)(2)(8)
1:	17/16	104.955	(34)
2:	9/8	203.910	(144)
3:	305/256	303.199	(610)
4:	5/4	386.314	(5)
5:	21/16	470.781	(21)
6:	89/64	570.880	(89)
7:	377/256	670.105	(377)
8:	3/2	701.955	(3)
9:	13/8	840.528	(13)
10:	55/32	937.632	(55)
11:	233/128	1037.023	(233)
12:	2/1	1200.000 octave	

This looked more promising. I also noticed that, due to the nature of interval addition, where ratios multiply against each other, that the numerators and denominators cancelled out, making the scale, in effect, a scale of harmonics numbered according to each of the elements of the fibonacci series.

For example: The first 15 elements of the fibonacci series are 1 1 2 3 5 8 13 21 34 55 89 144 233 377 610. Looking at our scale, we see 1/1 (for the ones) 2/1 (for the octave) 3/2 (3rd harmonic), 5/4 (5th harmonic expressed as a ratio within an octave), 8/1 = 2/1 = 1/1 (8th harmonic), 13/8 (13th harmonic), 21/16, 34/32=17/16 (34th harmonic expressed as a ratio within an octave) 55/32, 89/64, 144/128 = 72/64 = 36/32 = 9/8 (144th harmonic is an octave of the 9th harmonic...), etc.
So I then decided to take this scale out to 23 degrees and this is what I got. The last column is the number of the number series that the pitch is expressing. Another way of thinking about this is that the number of the series N becomes the Nth harmonic above the fundamental. (For reference, the first 23 unique elements of the fibonacci series are contained within: 1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 2584 4181 6765 10946 17711 28657 46368 75025 121393)

SCALE A (Just) :

0:	1/1	0.000	(1,2,8)
1:	4181/4096	35.559	(4181)
2:	17/16	104.955	(34)
3:	17711/16384	134.830	(17711)
4:	9/8	203.910	(144)
5:	75025/65536	234.101	(75025)
6:	305/256	303.199	(610)
7:	5/4	386.314	(5)
8:	323/256	402.468	(2584)
9:	21/16	470.781	(21)
10:	5473/4096	501.739	(10946)
11:	89/64	570.880	(89)
12:	1449/1024	601.010	(46368)
13:	377/256	670.105	(377)
14:	3/2	701.955	(3)
15:	1597/1024	769.378	(1597)
16:	13/8	840.528	(13)
17:	6765/4096	868.649	(6765)
18:	55/32	937.632	(55)
19:	28657/16384	967.920	(28657)
20:	233/128	1037.023	(233)
21:	121393/65536	1067.191	(121393)
22:	987/512	1136.288	(987)
23:	2/1	1200.000	

This seemed like a scale with many useable properties. In fact, Los Angeles composer Stephen James Taylor has been using it for many years. I wondered what the sequence of intervals made by the process of generating this scale would look like. I knew that it would start with a sequence of pure small-integer just ratios (3/2, 5/3, 8/5, etc), and that the intervals would get closer and closer to 833 cents, and that it would generate all the pitches in my scale, and that it would lead on and on to an infinite chain of intervals, but I was curious to see how the details of this sequence worked out:

Fibonacci Scale in Interval Order (first 10 intervals)

Int:		3/2	5/3	8/5	13/8	21/13
Pitch:	1/1	3/2	5/4	2/1	13/8	21/16
Cents:	0	702	386	1200	841	471
Scale Nr.	0	14	7	0 (23)	16	9

Int:	34/21	55/34	89/55	144/89	233/144
Pitch:	17/16	55/32	89/64	9/8	233/128
Cents:	105	938	571	204	1037
SclNr:.	2	18	11	4	20

Extending this out through the entire 23 note scale (listing only newly generated non-redundant pitches like the second 2/1 (1/1) in the above example) gave the following note order:

0 14 7 16 9 2 18 11 4 20 13 6 22 15 8 1 17 10 3 19 12 5 21

Looking at the intervals of the fibonacci series themselves, we can see how they do, indeed rapidly converge on the mean of 1.618033989...., which, expressed as a ratio over one, gives a value of 833.090297...cents:

Ratio:	3/2	5/3	8/5	13/8	21/13
Cents:	701.06	884.36	813.69	840.53	830.253

Ratio:	34/21	55/34	89/55	144/89	233/144
Cents:	834.18	832.60	833.23	833.11	833.08

This led me to speculate on what an equal tempered, or at least an equal-interval scale made up of intervals of 1.618...(approximately 833.090297c) would be. That is, if you added up, in a Pythagorean manner, 833.090297c intervals, what sort of a scale would you get? You would, in fact, get a scale that would be, for all practical purposes, identical to one made from the higher elements of the fibonacci series. The charm of the Just scale generated above, however, is that it DOES contain intervals which are more or less distant from the mean. The charm of the scale made from the upper elements of the series would be that it DOESN T contain intervals which deviate from the mean. You pays your money, and you takes your choice. Taking 23 intervals, to get a 23 note scale, like our fibonacci scale, yielded the following:

SCALE A (PYTHAGOREAN):

a chain of 23 intervals of 1.618..../1.000 made into cents

0:	1/1	0.000
1:	30.174 cents	30.174
2:	99.271 cents	99.271
3:	129.445 cents	129.445
4:	198.542 cents	198.542
5:	228.716 cents	228.716
6:	297.813 cents	297.813
7:	327.987 cents	327.987
8:	397.084 cents	397.084
9:	466.181 cents	466.181
10:	496.354 cents	496.354
11:	565.451 cents	565.451
12:	595.625 cents	595.625
13:	664.722 cents	664.722
14:	694.896 cents	694.896
15:	763.993 cent	763.993
16:	833.090 cents	833.090
17:	863.264 cents	863.264
18:	932.361 cents	932.361
19:	962.535 cents	962.535
20:	1031.632 cents	1031.632
21:	1061.806 cents	1061.806
22:	1130.903 cents	1130.903
23:	2/1	1200.000

The elements of the scale, taken in the order they were generated, were

0 16 9 2 18 11 4 20 13 6 22 15 8 1 17 10 3 19 12 5 21 14 7

So, taking the notes in the order they were generated gave a scale of 833.090 cent intervals.

I then began to realize, that as these were infinite series, they could indeed be extended infinitely. Was there any way, I wondered, to know when to stop , in order to get useable scales that had some sort of property of recognizability or coherence. I put this question to Erv Wilson, and he sent me a diagram of a metallophone he designed using the 23 note just scale given above, grouped into interlocking sets of 13 + 10 notes, in an analogous manner to the way the 7 note white key and 5 note black key scales are interlocked on the piano keyboard. He told me that the property known as Moments of Symmetry might provide some way of grouping these new scales into useable subsets.

A Moment of Symmetry (MOS) occurs when, in piling up a chain of equal intervals, a scale of two and only two melodic scale degree sizes results. There have been a number of articles on the workings of Moments of Symmetry (Wilson, 1975a, 1975b, Chalmers, 1975), but briefly, if we were making a chain of 3/2 perfect 5ths (701.955 cents), we would generate MOS scales with 2, 3, 5, 7, 12, 17, etc degrees in them. All the rest of the scales (for example 4, 6, 8, 9 scale degrees) will have 3 kinds of melodic scale degree intervals in them. Ignoring the trivial examples of scales of 2 and 3 scale degrees, the lowest example in the chain of 5ths scale where two MOS scales add up to a 3rd MOS scale with NO intervening examples of a MOS scale is the sequence of 5 scale degrees + 7 scale degrees = 12 scale degrees.

Examining the scales generated by piling up 833.090297 cent intervals, we find Moments of Symmetry with scales of 2, 3, 4, 7, 10, 13 23, etc. scale degrees. The lowest example of two scales that add up to a third with no intervening scales is 10 scale degrees + 13 scale degrees = 23 scale degrees. So if we divide our 23 tone just and Pythagorean scales into groups of 13 and 10 scale degrees we would get a scales interlocked in the following manners:

Just Fibonnacci scale divided up into 13 + 10 note groups:

Black Keys 1 3 5 8 10 12 15
White Keys 0 2 4 6 7 9 11 13 14 16
Black Keys 17 19 21
White Keys 18 20 22 0(23)

Pythagorean Fibonacci scale made from piling up 833.090297c intervals divided into 13 + 10 groups:

Black Keys 1 3 5 7 10 12 14
White Keys 0 2 4 6 8 9 11 13 15

Black Keys 17 19 21
White Keys 16 18 20 22 0 (23)

Although the breaks of 2 successive scale degrees occur at different places in the two scales, it should be obvious that the two structures simply transpositions of each other.

Intrigued by this, I generated Pythagorean and Just series for each of the succeeding 4 recurrent series, using the phenomena of the lowest two interlocking MOS scales that added up to a 3rd MOS scale as my guide for how many pitches my scales should have, and what structure they should have. For the next four scales the number of MOS degrees was as follows:

An = An-2 + An-1 (An/An-1 converges on 1.618033989...) MOS: 13 + 10 = 23
Bn = Bn-3 + Bn-1 (Bn/Bn-1 converges on 1.465571232...) MOS: 11 + 9 = 20
Cn = Cn-3 + Cn-2 (Cn/Cn-1 converges on 1.324717957...) MOS: 7 + 5 = 12
Dn = Dn-4 + Dn-1 (Dn/Dn-1 converges on 1.380277569...) MOS: 15 + 13 = 28
En = En-4 + En-3 (En/En-1 converges on 1.220744085...) MOS: 10 + 7 = 17.

Here are the scales generated from these sequences, with their divisions into the MOS groupings. In each case, the Just scale, made from the actual intervals of the number series are given, then the scale made from piling up the limit interval in a Pythagorean manner is given. In the Just Scales, the element of the number series a particular pitch is expressing is given in parentheses after the interval name.

SCALE B (JUST):

Just scale made from number series Bn=Bn-3+Bn-1. MOS division 11 + 9 = 20

(The series: 1 1 1 2 3 4 6 9 13 19 28 41 60 88 129 189 277 406 595 872 1278 1873 2745 4023 5896)

0:	1/1	0.000	(1)(2)(4)
1:	129/128	13.473	(129)
2:	277/256	136.491	(277)
3:	9/8	203.910	(9)
4:	595/512	260.095	(595)
5:	19/16	297.513	(19)
6:	639/512	383.607	(1278)
7:	41/32	429.062	(41)
8:	2745/2048	507.109	(2745)
9:	11/8	551.318	(88)
10:	737/512	630.625	(5896)
11:	189/128	674.691	(189)
12:	3/2	701.955	(3)
13:	203/128	798.403	(406)
14:	13/8	840.528	(13)
15:	109/64	921.821	(872)
16:	7/4	968.826	(28)
17:	1873/1024	1045.362	(1873)
18:	15/8	1088.269	(60)
19:	4023/2048	1168.867	(4023)
20:	2/1	1200.000	

Scale degrees generated in order:
0 12 3 14 5 16 7 18 9 1 11 2 13 4 15 6 17 8 19 10

Division into MOS 11 + 9 = 20:
Black Keys 2 4 6 8 10 13 15
White Keys 0 1 3 5 7 9 11 12 14 16
Black Keys 17 19
White Keys 18 0(20)

SCALE B (PYTHAGOREAN):

Pythagorean Scale Made of 1.465571232/1.0000, an interval of 661.755708c. MOS 11+9=20.

0:	1/1	0.000
1:	79.313 cents	79.313
2:	123.511 cents	123.511
3:	202.824 cents	202.824
4:	247.023 cents	247.023
5:	326.336 cents	326.336
6:	370.534 cents	370.534
7:	449.847 cents	449.847
8:	494.046 cents	494.046
9:	573.358 cents	573.358
10:	617.557 cents	617.557
11:	661.756 cents	661.756
12:	741.069 cents	741.069
13:	785.267 cents	785.267
14:	864.580 cents	864.580
15:	908.779 cents	908.779

16:	988.091 cents	988.091
17:	1032.290 cents	1032.290
18:	1111.603 cents	1111.603
19:	1155.801 cents	1155.801
20:	2/1	1200.000

Scale degrees generated in order:
0 11 2 13 4 15 6 17 8 19 10 1 12 3 14 5 16 7 18 9

Divsion into MOS 11 + 9 = 20
Black Keys 1 3 5 7 9 12 14 16
White Keys 0 2 4 6 8 10 11 13 15
Black Keys 18
White Keys 17 19 0(20)

SCALE C (JUST):

Just scale made from the number series Cn = Cn-3+Cn-2. MOS division 7 + 5 = 12

(The series: 1 0 1 1 1 2 2 3 4 5 7 9 12 16 21 28 37 49 65 86 114 151)

0:	1/1	0.000	(1)(2)(4)
1:	65/64	26.841	(65)
2:	9/8	203.910	(9)
3:	37/32	251.344	(37)
4:	151/128	286.086	(151)
5:	5/4	386.314	(5)
6:	21/16	470.781	(21)
7:	43/32	511.518	(86)
8:	3/2	701.955	(3)
9:	49/32	737.652	(49)
10:	7/4	968.826	(7)
11:	57/32	999.468	(114)
12:	2/1	1200.000	

Scale degrees generated in order:
0 8 5 10 2 6 3 9 1 7 11 4

Division into MOS: 7 + 5 = 12
Black Keys 1 4 7 9 11
White Keys 0 2 3 5 6 8 10 0(12)

SCALE C (PYTHAGOREAN):

Pythagorean scale made from ratios of 1.324717975/1.000 - 486.822277c MOS 7+5=12

0:	1/1	0.000
1:	34.111 cents	34.111
2:	68.223 cents	68.223
3:	260.467 cents	260.467
4:	294.578 cents	294.578
5:	486.822 cents	486.822
6:	520.934 cents	520.934
7:	555.045 cents	555.045
8:	747.289 cents	747.289
9:	781.400 cents	781.400
10:	973.645 cents	973.645
11:	1007.756 cents	1007.756
12:	2/1	1200.000

Scale degrees generated in order:
0 5 10 3 8 1 6 11 4 9 2 7

Division into MOS: 7+5 =12
Black Keys 2 4 7 9 11
White Keys 0 1 3 5 6 8 10 0(12)
(note that this is a different division than above)

SCALE D (JUST):

Just scale made from the number series Dn = Dn-4+Dn-1. MOS division 15 + 13 = 28
(The series:1 1 1 2 3 4 5 7 10 14 19 26 36 50 69 95 131 181 250 345 476 657 907 1252 1728 2385 3292 4544 6272 8657 11949 16493 22765 31422)

0:	1/1	0.000	(1)(2)(4)
1:	16493/16384	11.479	(16493)
2:	131/128	40.108	(131)
3:	8657/8192	95.582	(8657)
4:	69/64	130.229	(69)
5:	71/64	179.697	(4544)
6:	9/8	203.910	(36)
7:	2385/2048	263.728	(2385)
8:	19/16	297.513	(19)
9:	313/256	348.023	(1252)
10:	5/4	386.314	(5)
11:	657/512	431.699	(657)
12:	345/256	516.543	(345)
13:	22765/16384	569.436	(22765)
14:	181/128	599.815	(181)
15:	11949/8192	653.523	(11949)
16:	95/64	683.827	(95)
17:	3/2	701.955	(3)
18:	49/32	737.652	(6272)
19:	25/16	772.627	(50)
20:	823/512	821.698	(3292)
21:	13/8	840.528	(26)
22:	27/16	905.865	(1728)
23:	7/4	968.826	(7)(14)
24:	907/512	989.950	(907)
25:	119/64	1073.781	(476)
26:	15711/8192	1127.385	(31422)
27:	125/64	1158.941	(250)
28:	2/1	1200.000	

Scale degrees generated in order:
0 17 10 23 8 21 6 19 4 16 2 14 27 12 25 11 24 9 22 7 20 5 18 3 15 1 13 26

Division into MOS: 15+13=28
Black Keys 1 3 5 7 9 11 13 15
White Keys 0 2 4 6 8 10 12 14 16
Black Keys 18 20 22 24 26
White Keys 17 19 21 23 25 27 0

SCALE D (PYTHAGOREAN):

Pythagorean scale made from ratios of 1.380277569/1.000 - 557.950101c MOS 15+13=28

0:	1/1	0.000
1:	53.351 cents	53.351
2:	106.703 cents	106.703
3:	137.451 cents	137.451
4:	190.802 cents	190.802
5:	221.551 cents	221.551
6:	274.902 cents	274.902
7:	305.651 cents	305.651
8:	359.002 cents	359.002
9:	389.750 cents	389.750
10:	443.102 cents	443.102
11:	473.850 cents	473.850
12:	527.202 cents	527.202
13:	557.950 cents	557.950
14:	611.301 cents	611.301
15:	664.653 cents	664.653
16:	695.401 cents	695.401
17:	748.753 cents	748.753
18:	779.501 cents	779.501
19:	832.852 cents	832.852
20:	863.601 cents	863.601
21:	916.952 cents	916.952
22:	947.701 cents	947.701
23:	1001.052 cents	1001.052
24:	1031.800 cents	1031.800
25:	1085.152 cents	1085.152
26:	1115.900 cents	1115.900
27:	1169.252 cents	1169.252
28:	2/1	1200.000

Scale degrees generated in order:
0 13 26 11 24 9 22 7 20 5 18 3 16 1 14 27 12 25 10 23 8 21 6 19 4 17 2 15

Division into MOS: 15+13=28
Black Keys 2 4 6 8 10 12 15
White Keys 0 1 3 5 7 9 11 13 14 16
Black Keys 17 19 21 23 25 27
White Keys 18 20 22 24 26 0
(note again that this MOS division is different than the MOS division of the Just scale)

SCALE E (JUST):

Just scale made from the number series En = En-4+En-3. MOS division 10 + 7 = 17

(The series:1 0 0 1 1 0 1 2 1 1 3 3 2 4 6 5 6 10 11 11 16 21 22 27 37 43 49 64 80 92 113 144 172 205 257 316 377 462)

0:	1/1	0.000	(1,2,4, 16, 64)
1:	257/256	6.749	(257)
2:	9/8	203.910	(144)
3:	37/32	251.344	(37)
4:	79/64	364.537	(316)
5:	5/4	386.314	(5, 10)
6:	21/16	470.781	(21)
7:	43/32	511.518	(43,172)
8:	11/8	551.318	(11, 22)
9:	23/16	628.274	(92)
10:	377/256	670.105	(377)
11:	3/2	701.955	(3, 6)
12:	49/32	737.652	(49)
13:	205/128	815.376	(205)
14:	27/16	905.865	(27)
15:	113/64	984.215	(113)
16:	231/128	1022.099	(462)
17:	2/1	1200.000	

Scale degrees generated in order:
0 11 5 8 6 14 3 7 12 9 15 2 13 1 4 10 16

Division into MOS: 10+7=17
Black Keys 1 2 4 10 13 15 16
White Keys 0 3 5 6 7 8 9 11 12 14 0
(note the incredibly strange division of the scale here - maybe this is due to the very redundant nature of this series. The MOS division of the Pythagorean scale, below, is much more typical of MOS divisions.)

SCALE E (PYTHAGOREAN):

Pythagorean scale made from ratios of 1.220744085/1.000 - 345.312945c MOS 10+7=17

0:	1/1	0.000
1:	17.191 cents	17.191
2:	34.381 cents	34.381
3:	181.252 cents	181.252
4:	198.442 cents	198.442
5:	345.313 cents	345.313
6:	362.504 cents	362.504
7:	379.694 cents	379.694
8:	526.565 cents	526.565
9:	543.755 cents	543.755
10:	690.626 cents	690.626
11:	707.817 cents	707.817
12:	725.007 cents	725.007
13:	871.878 cents	871.878
14:	889.068 cents	889.068
15:	1035.939 cents	1035.939
16:	1053.129 cents	1053.129
17:	2/1	1200.000

Scale degrees generated in order:
0 5 10 15 3 8 13 1 6 11 16 4 9 14 2 7 12

Division into MOS: 10 + 7 = 17
Black Keys 2 4 7 9 12 14 16
White Keys 0 1 3 5 6 8 10 11 13 15 0(17)

I noticed that in generating each of the Just scales, that at a certain point, an interval was generated that was much less than a semitone away from one of the other tones of the scale. In the 12 note Pythagorean scale, made by stacking 702 cent fifths, after 12 fifths, we get a scale degree of 24 cents, which is very close to our starting point. This is called the Pythagorean comma , and one reason for stopping at 12 tones in this series is that that small interval was thought to be musically not useful. So for each of my series, I thought that I would see where the first narrow, or commatic interval occurred. For the first scale (Fibonacci series - Scale A), this yielded a 10 tone scale (the first comma was between 3/2 -702 c and 377/256 - 670 c)

Note:	1/1	17/16	9/8	5/4	21/16
Cents:	0	105	204	386	471

Note:	89/64	3/2	13/8	55/32	233/128
Cents:	571	702	841	938	1037

For Scale B a 9 tone scale resulted (comma between 1/1 - 0c and 129/128 - 13.5c):

Note:	1/1	9/8	19/16	41/32	11/8
Cents:	0	204	298	429	551

Note:	3/2	13/8	7/4	15/8
Cents:	702	841	969	1088

For Scale C a 7 tone scale resulted (comma between 3/2 - 702c and 49/32 - 737.7c):

Note:	1/1	9/8	37/32	5/4	21/16
Cents:	0	204	251	386	471

Note:	3/2	7/4
Cents:	702	969

For Scale D a 9 tone scale resulted (comma between 3/2 - 702c and 95/64 - 683.8c):

Note:	1/1	69/64	9/8	19/16	5/4
Cents:	0	130	204	298	386

Note:	3/2	25/16	13/8	7/4
Cents:	702	773	841	969

For Scale E a 7 (or 8) tone scale resulted (comma almost exactly halfway between 11/8 - 551.3c and 21/16 - 470.7c - which is 43/32 - 511.5c. Or, if a 40c comma is too large for you, the next interval in the series 49/43 - 737.6c is only 30 cents away from 3/2 - 702c. For practical purposes, though, the series of three tiny intervals in the scale below made by admitting the 43/32 might be thought redundant.)

Note:	1/1	37/32	5/4	21/16	(43/32)
Cents:	0	251	386	471	512

Note:	11/8	3/2	27/16
Cents:	551	702	906

Note that with the exception of scale D, all the scales have the smaller number of elements in their MOS division partners. That is for A, MOS13 + MOS 10 = MOS 23, and the commatically determined scale has 10 degrees. And for the exception, scale D, in the progression of MOS scales that leads eventually to the additive series MOS 15 + MOS 13 = MOS 28, we find that there is, indeed, a MOS with a scale of 9 scale degrees. These subsets of the Just scales might be very useful, especially for constructing modes based on these series.

In fact, Wilson, in his 2001 article, Pingala s Meru Prastara, and Sums of the Diagonals (Wilson, 2001) refers to scales resulting from the Second (Bn=Bn-3+Bn-1) and Third Recurrent Sequences respectively, as Meta-Pelog and Meta-Slendro . His Meta-Pelog is indeed a 9 note scale, encompassing elements 6 9 13 19 28 41 60 88 129 of the series. That is, his Meta-Pelog and my 9 tone scale above have only one difference - mine includes a 1/1 at 0 cents, and his scale eliminates this, substituting a 129/128 at 13.5 cents. Wilson is, of course, famous for his scales with no tonal centre (Rapoport, 1994), so his choice of series elements here is not surprising. His Meta-Slendro is, as he says, an unusual but compelling 12 tone scale made from elements of the 3^{rd} recurrent sequence (Cn=Cn-3+Cn-2) as follows: 9 12 16 21 28 37 49 65 86 114 151 200. Here, he goes one element further in the series than I do for his 12 tones, stopping at element 200. Further, because he starts at element 9, he eliminates element 5. So his 12 tone scale based on this series is identical to mine with the sole exception that there is no 5/4 at 386c, but there is, instead, a 25/16 at 773 cents.

Moreover, it is to be noted that these Just scales resulted from taking the elements of each series as members of the harmonic series. It would be just as feasible to take them as members of the subharmonic series, and get the inversions of the scales, and further, to then combine the harmonically generated and subharmonically generated scales to get symmetrical scales with very many melodic and harmonic possibilities. However, that might be a topic of exploration for another time. Already, with just the material explored, I ve generated between 10 and 35 new scales (depending on what you consider an independent scale, and what you consider just a subset of a larger scale). Wilson, has, to date, enumerated the first 192 of the Meru-Pascal recurrent sequences, of which I ve looked at only the first five. For the numerologically inclined, there would be an endless set of resources here to explore.

However, despite my fascination with the patterns in all the preceding, I m actually not one of the numerologically inclined. I m principally a composer, and my interest in all these things is in how they SOUND, what uses can be made of them, and how hearing these new relationships affects us, both emotionally and physically. I first wanted to hear how the scales sounded, in the order of generating their pitches. I thought that if a sequence of pitches as generated were heard canonically, we could also hear the harmonies produced by stacking the generating intervals of the scales. Using Emanuel op de Coul s Scala, and John Dunn s Softstep and Microtone, I proceeded to generate the following examples:

Example 1 is Scale A from the 1st recurrent sequence (the fibonacci sequence) played using the order of scale degrees as generated, with a piano timbre. First we hear the melody solo, on a pulse, and then with 2 voices, canonically, then 3 voices canonically, with a delay of one note between each of the voices. Note how at the end of the example, triadic type harmonies emerge from the result of the stackings and octave reductions of the 833 cent intervals.

Example 2 is the Pythagorean scale equivalent of Scale A (833 cents per generating step) treated in the same way. Here, the small-integer just intervals which characterize the beginning of Scale 1 are missing. Everything has the same quality, as it will in a Pythagorean scale. Notice that the harmonic quality of this scale is almost identical to the harmonic quality of the previous scale near the end of its sequence. That s because this scale is made exclusively of the interval the Just scale tends to as it progresses further and further through the series.

Example 3 is Just Scale B treated the same way. The mean interval here is 662 cents, a very sharp tritone, or an extremely flat fifth. Note the atonal type chords (sorry for the imprecise terminology!) that result from the canonic playing of this sequence as a canon.

Example 4 is the Pythagorean scale equivalent of Scale B (662 cents per generating step) treated in the same way. Again, note that the harmonic quality of this scale is almost identical with the harmonic quality of the just scale near the end of its sequence.

Example 5 is Just Scale C treated the same way. The mean interval here is 487 cents, a slightly flattened fourth, and indeed, at the end of the example, a three voice canonic playing of the sequence yields a series of fourth-like chords.

Example 6 is the Pythagorean scale equivalent of Scale D (487 cents per generating step) treated in the same manner.

Example 7 is Just Scale D treated the same way. The mean interval here is 558 cents, and the chords resulting from canonic playing of the sequence do indeed have a tritone quality, but to my ear sound less dissonant than the chords resulting from the canonic playing of Just Scale B.

Example 8 is the Pythagorean scale equivalent of Scale D (558 cents per generating step).

Example 9 is Just Scale E treated the same way. The mean interval here is 345 cents, which is a neutral third. Chords made from stacking these have a neutral triad quality. However, with the larger Just intervals at the start of the sequence, and the fact that this series takes a little longer to settle down into intervals around its mean, this harmonic quality is only heard very briefly at the end of the series.

Example 10 is the Pythagorean scale equivalent of Scale E (345 cents per generating step). Here, the neutral triad quality implied by Just Scale E is heard much more clearly.

So after all this work on scales, and hearing some examples of the quality of harmonies implied by some of their aspect, it s now time to begin to compose with these materials. Questions about pitch organization naturally occur, but additional questions come to mind at this point about timbre (could these series generate interesting spectra, or if we use simple sine waves to play these scales, will the inherent harmonies of the scales and the pure sine waves combine to make additive synthesis timbres on their own?), rhythm (without getting bogged down in serialism or fibonacciana, can interesting rhythmic sequences be made with these proportions?), and spatiality (can these proportions be used to structure space as well as time? Or, again, if we simply use sine wave timbres, will the proportions of the scale then naturally interact with the acoustics of the space the sounds are played in to create an emergent spatial architecture determined by the wavelengths of the pitches in conjunction with the dimensions of the performing space?). Clearly, after all this work on number series and proportion, we are actually only at the beginning of a sonic adventure exploring these materials.

A first exploration of these scales was made in the compositions grouped under the title of The Mossy Slopes of Mt. Meru. In the first of this series, a series of imaginary heterophonic duets for sampled flute and two sampled harp lines was made. Each piece was in one of the complete Just Scales, A - E, described earlier. In each piece, the three melodies (one flute and two harp) were assembled by using the sequence of generating intervals for that scale as a kind of row. The program moved back and forth through the row by steps of 1, 2, 3 or other number of steps, based on what numbers were in the corresponding number series. Durations for each of the three simultaneous melodies were also drawn from numbers available in the number series (a random choice from a small set of durations was used here), and dynamics were chosen in the same way. The character of each melody then, was determined by proportions inherent in the number series that generated the scale. Over the course of this short piece (five movements of 90 seconds each) different characters of harmony, tempo and gesture can clearly be heard. These characters are an emergent property of the different characters of each number series. As an example, here are the beginnings of three of the pieces:

Examples 11, 12, 13: The first 20" each of Just Scale A , Just Scale B and Just Scale C from The Mossy Slopes of Mt. Meru - II: Imaginary Flute and Harp in a Virtual Garage.

A more thorough exploration of the harmonies available in the MOS divisions of these scales was made in the hour long composition The Mossy Slopes of Mt. Meru - I: Architectural Chords. In this piece, each of the 30 possible scales outlined in this paper is used in a two minute section. They are played with pure sine waves, over a 3, 4 or 5 octave range, with chords that can have between 2 and 10 elements. Melodic motion through the scales was step-wise, in numbers of scales steps determined by low numbers of the relevant number series. For example, there could be steps of 1, 2, 3, 5, or 8 scale degrees in the scales derived from the Fibonacci series. The exact number of steps was chosen by a shift register feedback algorithm I first used in 1972, now embodied in software. (Burt, 1975) Similarly, dynamics and durations were also chosen in a similar manner from lists of numbers taken from the relevant number series. The results of using pure sine waves were what I had hoped for. The sine waves at times did indeed add up to make composite timbres, especially when a wide range of dynamics was chosen for a chord, at times sounding like a simple interval or chord, but at other times fusing into some quite delicious and complex timbres. When I used timbres with more harmonics, this fusing effect was nowhere near as pronounced as when I used pure sines, so I chose to stay with them. Further, because sine waves carve any performance space up into areas of high and low pressure, each chord created quite remarkable spatial effects. These are heard most clearly by moving one s head slightly while listening to these chords. Single pitches, and sometimes whole ranges of the chords seem to appear and disappear as one s head moves. This clearly demonstrates that each chord is indeed dividing the performing space up into a sonic architecture proportioned by the ratios of the pitches in each chord. Sine waves are notoriously non-directional, but just for a little extra dimensionality, I also panned each sine wave slightly in space. In most spaces this seems to make no difference, but when heard over headphones, the effect of this panning is quite marked. Again, when I tried this with more complex timbres, the effect was not as pronounced as when I used pure sines. Since I wanted this piece to be as pure an exposition of the qualities of these scales as I could get, I decided to stay with the sine wave timbres. In live performance, I control the tempo of the chords, how many notes make a chord, and which scale is being used. Pitch, durations and dynamics are chosen algorithmically. I find this piece very beautiful, and I think I can safely say that a large measure of the beauty is the result of emergent properties of these scales and number series being allowed to reveal themselves as a result of my processes and algorithms.

Examples 14 & 15: 30 second excerpts from 1: Just Scale A: 13/23 MOS and 23: Pythagorean Scale D: 28/28 MOS from The Mossy Steps of Mt. Meru - I: Architectural Chords showing the different kinds of harmonies, timbres, and spatialities available from these scales.

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