MATH REVIEW
for
Practicing to Take the
GRE®
General Test

Published for the Graduate Record Examinations Board
by Educational Testing Service

MATH REVIEW

The Math Review is designed to familiarize you with the mathematical skills and concepts likely to be tested on the Graduate Record Examinations General Test. This material, which is divided into the four basic content areas of arithmetic, algebra, geometry, and data analysis, includes many definitions and examples with solutions, and there is a set of exercises (with answers) at the end of each of these four sections. Note, however, this review is not intended to be comprehensive. It is assumed that certain basic concepts are common knowledge to all examinees. Emphasis is, therefore, placed on the more important skills, concepts, and definitions, and on those particular areas that are frequently confused or misunderstood. If any of the topics seem especially unfamiliar, we encourage you to consult appropriate mathematics texts for a more detailed treatment of those topics.

TABLE OF CONTENTS

ARITHMETIC

1.1 Integers

The set of *integers*, I, is composed of all the counting numbers (i.e., 1, 2, 3, . . .), zero, and the negative of each counting number; that is,

$$I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Therefore, some integers are *positive*, some are *negative*, and the integer 0 is neither positive nor negative. Integers that are multiples of 2 are called *even integers*, namely $\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$. All other integers are called *odd integers*; therefore $\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$ represents the set of all odd integers. Integers in a sequence such as 57, 58, 59, 60, or −14, −13, −12, −11 are called *consecutive* integers.

The rules for performing basic arithmetic operations with integers should be familiar to you. Some rules that are occasionally forgotten include:

(i) Multiplication by 0 always results in 0; e.g., $(0)(15) = 0$.

(ii) Division by 0 is not defined; e.g., $5 \div 0$ has no meaning.

(iii) Multiplication (or division) of two integers with different signs yields a negative result; e.g., $(-7)(8) = -56$ and $(-12) \div (4) = -3$.

(iv) Multiplication (or division) of two *negative* integers yields a positive result; e.g., $(-5)(-12) = 60$ and $(-24) \div (-3) = 8$.

The division of one integer by another yields either a zero remainder, sometimes called "dividing evenly," or a positive-integer remainder. For example, 215 divided by 5 yields a zero remainder, but 153 divided by 7 yields a remainder of 6.

$$\begin{array}{r} 43 \\ 5\overline{)215} \\ \underline{20} \\ 15 \\ \underline{15} \\ 0 = \text{Remainder} \end{array} \qquad \qquad \begin{array}{r} 21 \\ 7\overline{)153} \\ \underline{14} \\ 13 \\ \underline{7} \\ 6 = \text{Remainder} \end{array}$$

When we say that an integer N is *divisible by* an integer x, we mean that N divided by x yields a zero remainder.

The multiplication of two integers yields a third integer. The first two integers are called *factors*, and the third integer is called the *product*. The product is said to be a *multiple* of both factors, and it is also *divisible* by both factors. Therefore, since $(2)(7) = 14$, we can say that

2 and 7 are factors and 14 is the product,
14 is a multiple of both 2 and 7,
and 14 is divisible by both 2 and 7.

Whenever an integer N is divisible by an integer x, we say that x is a *divisor* of N. For the set of positive integers, any integer N that has exactly two distinct positive divisors, 1 and N, is said to be a *prime number*. The first ten prime numbers are

2, 3, 5, 7, 11, 13, 17, 19, 23, and 29.

The integer 14 is not a prime number because it has four divisors: 1, 2, 7, and 14. The integer 1 is not a prime number because it has only one positive divisor.

1.2 Fractions

A *fraction* is a number of the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. The a is called the *numerator* of the fraction, and b is called the *denominator*. For example, $\frac{-7}{5}$ is a fraction that has -7 as its numerator and 5 as its denominator. Since the fraction $\frac{a}{b}$ means $a \div b$, b cannot be zero. If the numerator and denominator of the fraction $\frac{a}{b}$ are both multiplied by the same integer, the resulting fraction will be equivalent to $\frac{a}{b}$. For example,

$$\frac{-7}{5} = \frac{(-7)(4)}{(5)(4)} = \frac{-28}{20}.$$

This technique comes in handy when you wish to add or subtract fractions.

To add two fractions with the same denominator, you simply add the numerators and keep the denominator the same.

$$\frac{-8}{11} + \frac{5}{11} = \frac{-8+5}{11} = \frac{-3}{11}$$

If the denominators are *not* the same, you may apply the technique mentioned above to make them the same before doing the addition.

$$\frac{5}{12} + \frac{2}{3} = \frac{5}{12} + \frac{(2)(4)}{(3)(4)} = \frac{5}{12} + \frac{8}{12} = \frac{5+8}{12} = \frac{13}{12}$$

The same method applies for subtraction.

To multiply two fractions, multiply the two numerators and multiply the two denominators (the denominators need not be the same).

$$\left(\frac{10}{7}\right)\left(\frac{-1}{3}\right) = \frac{(10)(-1)}{(7)(3)} = \frac{-10}{21}$$

To divide one fraction by another, first *invert* the fraction you are dividing by, and then proceed as in multiplication.

$$\frac{17}{8} \div \frac{3}{5} = \left(\frac{17}{8}\right)\left(\frac{5}{3}\right) = \frac{(17)(5)}{(8)(3)} = \frac{85}{24}$$

An expression such as $4\frac{3}{8}$ is called a *mixed fraction*; it means $4 + \frac{3}{8}$. Therefore,

$$4\frac{3}{8} = 4 + \frac{3}{8} = \frac{32}{8} + \frac{3}{8} = \frac{35}{8}.$$

1.3 Decimals

In our number system, all numbers can be expressed in decimal form using base 10. A decimal point is used, and the place value for each digit corresponds to a power of 10, depending on its position relative to the decimal point. For example, the number 82.537 has 5 digits, where

"8" is the "tens" digit; the place value for "8" is 10.

"2" is the "units" digit; the place value for "2" is 1.

"5" is the "tenths" digit; the place value for "5" is $\frac{1}{10}$.

"3" is the "hundredths" digit; the place value for "3" is $\frac{1}{100}$.

"7" is the "thousandths" digit; the place value for "7" is $\frac{1}{1000}$.

Therefore, 82.537 is a short way of writing

$$(8)(10) + (2)(1) + (5)\left(\frac{1}{10}\right) + (3)\left(\frac{1}{100}\right) + (7)\left(\frac{1}{1000}\right), \text{ or}$$

$$80 + 2 + 0.5 + 0.03 + 0.007.$$

This numeration system has implications for the basic operations. For addition and subtraction, you must always remember to line up the decimal points:

$$\begin{array}{r} 126.5 \\ +\ 68.231 \\ \hline 194.731 \end{array} \qquad \begin{array}{r} 126.5 \\ -\ 68.231 \\ \hline 58.269 \end{array}$$

To multiply decimals, it is not necessary to align the decimal points. To determine the correct position for the decimal point in the product, you simply add the number of digits to the right of the decimal points in the decimals being multiplied. This sum is the number of decimal places required in the product.

$$\begin{array}{rl} 15.381 & \text{(3 decimal places)} \\ \times\ \ .14 & \text{(2 decimal places)} \\ \hline 61524 & \\ 15381\ & \\ \hline 2.15334 & \text{(5 decimal places)} \end{array}$$

To divide a decimal by another, such as $62.744 \div 1.24$, or

$$1.24\overline{)62.744},$$

first move the decimal point in the divisor to the right until the divisor becomes an integer, then move the decimal point in the dividend the same number of places;

$$124\overline{)6274.4}$$

This procedure determines the correct position of the decimal point in the quotient (as shown). The division can then proceed as follows:

$$
\begin{array}{r}
50.6 \\
124\overline{)6274.4} \\
\underline{620} \\
744 \\
\underline{744} \\
0
\end{array}
$$

Conversion from a given decimal to an equivalent fraction is straightforward. Since each place value is a power of ten, every decimal can be converted easily to an integer divided by a power of ten. For example,

$$84.1 = \frac{841}{10}$$

$$9.17 = \frac{917}{100}$$

$$0.612 = \frac{612}{1000}$$

The last example can be reduced to lowest terms by dividing the numerator and denominator by 4, which is their *greatest common factor*. Thus,

$$0.612 = \frac{612}{1000} = \frac{612 \div 4}{1000 \div 4} = \frac{153}{250} \text{ (in lowest terms).}$$

Any fraction can be converted to an equivalent decimal. Since the fraction $\frac{a}{b}$ means $a \div b$, we can divide the numerator of a fraction by its denominator to convert the fraction to a decimal. For example, to convert $\frac{3}{8}$ to a decimal, divide 3 by 8 as follows.

$$
\begin{array}{r}
0.375 \\
8\overline{)3.000} \\
\underline{2\,4} \\
60 \\
\underline{56} \\
40 \\
\underline{40} \\
0
\end{array}
$$

1.4 Exponents and Square Roots

Exponents provide a shortcut notation for repeated multiplication of a number by itself. For example, "3^4" means (3)(3)(3)(3), which equals 81. So, we say that $3^4 = 81$; the "4" is called an *exponent* (or power). The exponent tells you how many factors are in the product. For example,

$$2^5 = (2)(2)(2)(2)(2) = 32$$
$$10^6 = (10)(10)(10)(10)(10)(10) = 1{,}000{,}000$$
$$(-4)^3 = (-4)(-4)(-4) = -64$$
$$\left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{16}$$

When the exponent is 2, we call the process *squaring*. Therefore, "5^2" can be read "5 squared."

Exponents can be negative or zero, with the following rules for any nonzero number m.

$$m^0 = 1$$
$$m^{-1} = \frac{1}{m}$$
$$m^{-2} = \frac{1}{m^2}$$
$$m^{-3} = \frac{1}{m^3}$$
$$m^{-n} = \frac{1}{m^n} \text{ for all integers } n.$$

If $m = 0$, then these expressions are not defined.

A *square root* of a positive number N is a real number which, when squared, equals N. For example, a square root of 16 is 4 because $4^2 = 16$. Another square root of 16 is –4 because $(-4)^2 = 16$. In fact, all positive numbers have two square roots that differ only in sign. The square root of 0 is 0 because $0^2 = 0$. Negative numbers do *not* have square roots because the square of a real number cannot be negative. If $N > 0$, then the positive square root of N is represented by $\sqrt{N}$, read "radical N." The negative square root of N, therefore, is represented by $-\sqrt{N}$.

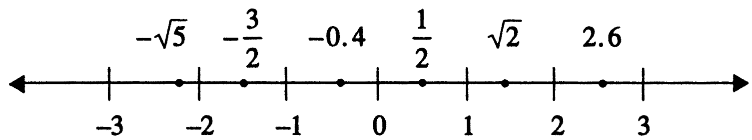
Two important rules regarding operations with radicals are:
If $a > 0$ and $b > 0$, then

(i) $\left(\sqrt{a}\right)\left(\sqrt{b}\right) = \sqrt{ab}$; e.g., $\left(\sqrt{5}\right)\left(\sqrt{20}\right) = \sqrt{100} = 10$

(ii) $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$; e.g., $\frac{\sqrt{192}}{\sqrt{4}} = \sqrt{48} = \sqrt{(16)(3)} = \left(\sqrt{16}\right)\left(\sqrt{3}\right) = 4\sqrt{3}$

1.5 Ordering and the Real Number Line

The set of all *real numbers*, which includes all integers and all numbers with values between them, such as 1.25, $\frac{2}{3}$, $\sqrt{2}$, etc., has a natural ordering, which can be represented by the *real number line*:



Every real number corresponds to a point on the real number line (see examples shown above). The real number line is infinitely long in both directions.

For any two numbers on the real number line, the number to the left is *less than* the number to the right. For example,

$$-\sqrt{5} < -\frac{3}{2}$$
$$-1.75 < \sqrt{2}$$
$$\frac{5}{2} < 7.1$$

Since $2 < 5$, it is also true that 5 is greater than 2, which is written "5 > 2."
If a number N is *between* 1.5 and 2 on the real number line, you can express that fact as $1.5 < N < 2$.

1.6 Percent

The term *percent* means *per hundred* or *divided by one hundred*. Therefore,

$$43\% = \frac{43}{100} = 0.43$$
$$300\% = \frac{300}{100} = 3$$
$$0.5\% = \frac{0.5}{100} = 0.005$$

To find out what 30% of 350 is, you multiply 350 by either 0.30 or $\frac{30}{100}$,

$$30\% \text{ of } 350 = (350)(0.30) = 105$$

or

$$30\% \text{ of } 350 = (350)\left(\frac{30}{100}\right) = (350)\left(\frac{3}{10}\right) = \frac{1{,}050}{10} = 105.$$

To find out what percent of 80 is 5, you set up the following equation and solve for x:

$$\frac{5}{80} = \frac{x}{100}$$

$$x = \frac{500}{80} = 6.25$$

So 5 is 6.25% of 80. The number 80 is called the *base* of the percent. Another way to view this problem is to simply divide 5 by the base, 80, and then multiply the result by 100 to get the percent.

If a quantity *increases* from 600 to 750, then the *percent increase* is found by dividing the amount of increase, 150, by the base, 600, which is the first (or the smaller) of the two given numbers, and then multiplying by 100:

$$\left(\frac{150}{600}\right)(100)\% = 25\%.$$

If a quantity *decreases* from 500 to 400, then the *percent decrease* is found by dividing the amount of decrease, 100, by the base, 500, which is the first (or the larger) of the two given numbers, and then multiplying by 100:

$$\left(\frac{100}{500}\right)(100)\% = 20\%.$$

Other ways to state these two results are "750 is 25 *percent greater than* 600" and "400 is 20 *percent less than* 500."

In general, for any positive numbers x and y, where $x < y$,

$$y \text{ is } \left(\frac{y - x}{x}\right)(100) \text{ percent greater than } x$$

$$x \text{ is } \left(\frac{y - x}{y}\right)(100) \text{ percent less than } y$$

Note that in each of these statements, the base of the percent is in the denominator.

1.7 Ratio

The ratio of the number 9 to the number 21 can be expressed in several ways; for example,

9 to 21

9:21

$$\frac{9}{21}$$

Since a ratio is in fact an implied division, it can be reduced to lowest terms. Therefore, the ratio above could also be written:

3 to 7

3:7

$$\frac{3}{7}$$

1.8 Absolute Value

The *absolute value* of a number N, denoted by $|N|$, is defined to be N if N is positive or zero and $-N$ if N is negative. For example,

$$\left|\frac{1}{2}\right| = \frac{1}{2},\ |0| = 0, \text{ and } |-2.6| = -(-2.6) = 2.6.$$

Note that the absolute value of a number cannot be negative.

ARITHMETIC EXERCISES

(Answers on pages 17 and 18)

1. Evaluate:

   (a) $15 - (6 - 4)(-2)$
   (b) $(2 - 17) \div 5$
   (c) $(60 \div 12) - (-7 + 4)$
   (d) $(3)^4 - (-2)^3$
   (e) $(-5)(-3) - 15$
   (f) $(-2)^4(15 - 18)^4$
   (g) $(20 \div 5)^2(-2 + 6)^3$
   (h) $(-85)(0) - (-17)(3)$

2. Evaluate:

   (a) $\frac{1}{2} - \frac{1}{3} + \frac{1}{12}$
   (b) $\left(\frac{3}{4} + \frac{1}{7}\right)\left(\frac{-2}{5}\right)$
   (c) $\left(\frac{7}{8} - \frac{4}{5}\right)^2$
   (d) $\left(\frac{3}{-8}\right) \div \left(\frac{27}{32}\right)$

3. Evaluate:

   (a) $12.837 + 1.65 - 0.9816$
   (b) $100.26 \div 1.2$
   (c) $(12.4)(3.67)$
   (d) $(0.087)(0.00021)$

4. State for each of the following whether the answer is an *even* integer or an *odd* integer.

   (a) The sum of two even integers
   (b) The sum of two odd integers
   (c) The sum of an even integer and an odd integer
   (d) The product of two even integers
   (e) The product of two odd integers
   (f) The product of an even integer and an odd integer

5. Which of the following integers are divisible by 8 ?

   (a) 312 (b) 98 (c) 112 (d) 144

6. List all of the positive divisors of 372.

7. Which of the divisors found in #6 are prime numbers?

8. Which of the following integers are prime numbers?
   19, 2, 49, 37, 51, 91, 1, 83, 29

9. Express 585 as a product of prime numbers.

10. Which of the following statements are true?

(a) $-5 < 3.1$

(b) $\sqrt{16} = 4$

(c) $7 \div 0 = 0$

(d) $0 < |-1.7|$

(e) $0.3 < \frac{1}{3}$

(f) $(-1)^{87} = -1$

(g) $\sqrt{9} < 0$

(h) $\frac{21}{28} = \frac{3}{4}$

(i) $-|-23| = 23$

(j) $\frac{1}{2} > \frac{1}{17}$

(k) $(59)^3(59)^2 = (59)^6$

(l) $-\sqrt{25} < -4$

11. Perform the indicated operations.

(a) $5\sqrt{3} + \sqrt{27}$

(b) $(\sqrt{6})(\sqrt{30})$

(c) $(\sqrt{300}) \div (\sqrt{12})$

(d) $(\sqrt{5})(\sqrt{2}) - \sqrt{90}$

12. Express the following percents in decimal form and in fraction form (in lowest terms).

(a) 15% (b) 27.3% (c) 131% (d) 0.02%

13. Express each of the following as a percent.

(a) 0.8 (b) 0.197 (c) 5.2 (d) $\frac{3}{8}$ (e) $2\frac{1}{2}$ (f) $\frac{3}{50}$

14. Find:

(a) 40% of 15

(b) 150% of 48

(c) 0.6% of 800

(d) 8% of 5%

15. If a person's salary increases from \$200 per week to \$234 per week, what is the percent increase?

16. If an athlete's weight decreases from 160 pounds to 152 pounds, what is the percent decrease?

17. A particular stock is valued at $40 per share. If the value increases 20 percent and then decreases 25 percent, what is the value of the stock per share after the decrease?

18. Express the ratio of 16 to 6 three different ways in lowest terms.

19. If the ratio of men to women on a committee of 20 members is 3 to 2, how many members of the committee are women?

ANSWERS TO ARITHMETIC EXERCISES

1. (a) 19
   (b) -3
   (c) 8
   (d) 89
   (e) 0
   (f) 1,296
   (g) 1,024
   (h) 51

2. (a) $\frac{1}{4}$
   (b) $-\frac{5}{14}$
   (c) $\frac{9}{1,600}$
   (d) $-\frac{4}{9}$

3. (a) 13.5054
   (b) 83.55
   (c) 45.508
   (d) 0.00001827

4. (a) even
   (b) even
   (c) odd
   (d) even
   (e) odd
   (f) even

5. (a), (c), and (d)

6. 1, 2, 3, 4, 6, 12, 31, 62, 93, 124, 186, 372

7. 2, 3, 31

8. 19, 2, 37, 83, 29

9. (3)(3)(5)(13)

10. (a), (b), (d), (e), (f), (h), (j), (l)

11. (a) $8\sqrt{3}$
    (b) $6\sqrt{5}$
    (c) 5
    (d) $-2\sqrt{10}$

12. (a) 0.15, $\frac{3}{20}$
    (b) 0.273, $\frac{273}{1,000}$
    (c) 1.31, $\frac{131}{100}$
    (d) 0.0002, $\frac{1}{5,000}$

13. (a) 80%
    (b) 19.7%
    (c) 520%
    (d) 37.5%
    (e) 250%
    (f) 6%

14. (a) 6
    (b) 72
    (c) 4.8
    (d) 0.004

15. 17%

16. 5%

17. $36

18. 8 to 3, 8:3, $\frac{8}{3}$

19. 8

ALGEBRA

2.1 Translating Words into Algebraic Expressions

Basic algebra is essentially advanced arithmetic; therefore much of the terminology and many of the rules are common to both areas. The major difference is that in algebra variables are introduced, which allows us to solve problems using equations and inequalities.

If the square of the number x is multiplied by 3, and then 10 is added to that product, the result can be represented by $3x^2 + 10$. If John's present salary S is increased by 14 percent, then his new salary is $1.14S$. If y gallons of syrup are to be distributed among 5 people so that one particular person gets 1 gallon and the rest of the syrup is divided equally among the remaining 4, then each of these 4 people will get $\frac{y-1}{4}$ gallons of syrup. Combinations of letters (variables) and numbers such as $3x^2 + 10$, $1.14S$, and $\frac{y-1}{4}$ are called *algebraic expressions*.

One way to work with algebraic expressions is to think of them as *functions*, or "machines," that take an input, say a value of a variable x, and produce a corresponding output. For example, in the expression $\frac{2x}{x-6}$, the input $x = 1$ produces the corresponding output $\frac{2(1)}{1-6} = -\frac{2}{5}$. In function notation, the expression $\frac{2x}{x-6}$ is called a function and is denoted by a letter, often the letter f or g, as follows:

$$f(x) = \frac{2x}{x-6}.$$

We say that this equation *defines* the function f. For this example with input $x = 1$ and output $-\frac{2}{5}$, we write $f(1) = -\frac{2}{5}$. The output $-\frac{2}{5}$ is called the *value of the function* corresponding to the input $x = 1$. The value of the function corresponding to $x = 0$ is 0, since

$$f(0) = \frac{2(0)}{0-6} = -\frac{0}{6} = 0.$$

In fact, any real number x can be used as an input value for the function f, except for $x = 6$, as this substitution would result in a division by 0. Since $x = 6$ is not a valid input for f, we say that f is not defined for $x = 6$.

As another example, let h be the function defined by

$$h(z) = z^2 + \sqrt{z} + 3.$$

Note that $h(0) = 3$, $h(1) = 5$, $h(10) = 103 + \sqrt{10} \approx 106.2$, but $h(-10)$ is not defined since $\sqrt{-10}$ is not a real number.

2.2 Operations with Algebraic Expressions

Every algebraic expression can be written as a single term or a series of terms separated by plus or minus signs. The expression $3x^2 + 10$ has two terms; the expression $1.14S$ is a single term; the expression $\frac{y-1}{4}$, which can be written $\frac{y}{4} - \frac{1}{4}$, has two terms. In the expression $2x^2 + 7x - 5$, 2 is the *coefficient* of the x^2 *term*, 7 is the coefficient of the x *term*, and −5 is the *constant term*.

The same rules that govern operations with numbers apply to operations with algebraic expressions. One additional rule, which helps in simplifying algebraic expressions, is that terms with the same variable part can be combined. Examples are:

$$2x + 5x = (2 + 5)x = 7x$$
$$x^2 - 3x^2 + 6x^2 = (1 - 3 + 6)x^2 = 4x^2$$
$$3xy + 2x - xy - 3x = (3 - 1)xy + (2 - 3)x = 2xy - x$$

Any number or variable that is a factor of each term in an algebraic expression can be factored out. Examples are:

$$4x + 12 = 4(x + 3)$$
$$15y^2 - 9y = 3y(5y - 3)$$
$$\frac{7x^2 + 14x}{2x + 4} = \frac{7x(x + 2)}{2(x + 2)} = \frac{7x}{2} \quad (\text{if } x \neq -2)$$

Another useful tool for factoring algebraic expressions is the fact that $a^2 - b^2 = (a + b)(a - b)$. For example,

$$\frac{x^2 - 9}{4x - 12} = \frac{(x + 3)(x - 3)}{4(x - 3)} = \frac{x + 3}{4} \ (\text{if } x \neq 3).$$

To multiply two algebraic expressions, each term of the first expression is multiplied by each term of the second, and the results are added. For example,

$$\begin{aligned}(x + 2)(3x - 7) &= x(3x) + x(-7) + 2(3x) + 2(-7)\\ &= 3x^2 - 7x + 6x - 14\\ &= 3x^2 - x - 14\end{aligned}$$

A statement that equates two algebraic expressions is called an *equation*. Examples of equations are:

$$3x + 5 = -2 \quad \text{(linear equation in one variable)}$$
$$x - 3y = 10 \quad \text{(linear equation in two variables)}$$
$$20y^2 + 6y - 17 = 0 \quad \text{(quadratic equation in one variable)}$$

2.3 Rules of Exponents

Some of the basic rules of exponents are:

(a) $x^{-a} = \frac{1}{x^a}$ $(x \neq 0)$

Example: $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$.

(b) $(x^a)(x^b) = x^{a+b}$

Example: $(3^2)(3^4) = 3^{2+4} = 3^6 = 729$.

(c) $(x^a)(y^a) = (xy)^a$

Example: $(2^3)(3^3) = 6^3 = 216$.

(d) $\frac{x^a}{x^b} = x^{a-b} = \frac{1}{x^{b-a}}$ $(x \neq 0)$

Examples: $\frac{5^7}{5^4} = 5^{7-4} = 5^3 = 125$ and $\frac{4^3}{4^8} = \frac{1}{4^{8-3}} = \frac{1}{4^5} = \frac{1}{1{,}024}$.

(e) $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$ $(y \neq 0)$

Example: $\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$.

(f) $(x^a)^b = x^{ab}$

Example: $(2^5)^2 = 2^{10} = 1{,}024$.

(g) If $x \neq 0$, then $x^0 = 1$.

Examples: $7^0 = 1$; $(-3)^0 = 1$; 0^0 is not defined.

2.4 Solving Linear Equations

(a) One variable.

To solve a linear equation in one variable means to find the value of the variable that makes the equation true. Two equations that have the same solution are said to be *equivalent*. For example, $x + 1 = 2$ and $2x + 2 = 4$ are equivalent equations; both are true when $x = 1$ and are false otherwise.

Two basic rules are important for solving linear equations.

(i) When the same constant is added to (or subtracted from) both sides of an equation, the equality is preserved, and the new equation is equivalent to the original.

(ii) When both sides of an equation are multiplied (or divided) by the same nonzero constant, the equality is preserved, and the new equation is equivalent to the original.

For example,

$$\begin{aligned} 3x - 4 &= 8 \\ 3x - 4 + 4 &= 8 + 4 \quad \text{(4 added to both sides)} \\ 3x &= 12 \\ \frac{3x}{3} &= \frac{12}{3} \quad \text{(both sides divided by 3)} \\ x &= 4 \end{aligned}$$

(b) Two variables.

To solve linear equations in two variables, it is necessary to have two equations that are not equivalent. To solve such a "system" of simultaneous equations, e.g.,

$$\begin{aligned} 4x + 3y &= 13 \\ x + 2y &= 2 \end{aligned}$$

there are two basic methods. In the *first method*, you use either equation to express one variable in terms of the other. In the system above, you could express x in the second equation in terms of y (i.e., $x = 2 - 2y$), and then substitute $2 - 2y$ for x in the first equation to find the solution for y:

$$\begin{aligned} 4(2 - 2y) + 3y &= 13 \\ 8 - 8y + 3y &= 13 \\ -8y + 3y &= 5 \quad \text{(8 subtracted from both sides)} \\ -5y &= 5 \quad \text{(terms combined)} \\ y &= -1 \quad \text{(both sides divided by } -5\text{)} \end{aligned}$$

Then -1 can be substituted for y in the second equation to solve for x:

$$\begin{aligned} x + 2y &= 2 \\ x + 2(-1) &= 2 \\ x - 2 &= 2 \\ x &= 4 \quad \text{(2 added to both sides)} \end{aligned}$$

In the *second method*, the object is to make the coefficients of one variable the same in both equations so that one variable can be eliminated by either adding both equations together or subtracting one from the other. In the same example, both sides of the second equation could be multiplied by 4, yielding $4(x + 2y) = 4(2)$, or $4x + 8y = 8$. Now we have two equations with the same x coefficient:

$$\begin{aligned} 4x + 3y &= 13 \\ 4x + 8y &= 8 \end{aligned}$$

If the second equation is subtracted from the first, the result is $-5y = 5$. Thus, $y = -1$, and substituting -1 for y in either one of the original equations yields $x = 4$.

2.5 Solving Quadratic Equations in One Variable

A *quadratic* equation is any equation that can be expressed as $ax^2 + bx + c = 0$, where a, b, and c are real numbers ($a \neq 0$). Such an equation can always be solved by the *formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For example, in the quadratic equation $2x^2 - x - 6 = 0$, $a = 2$, $b = -1$, and $c = -6$. Therefore, the formula yields

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-6)}}{2(2)} \\ &= \frac{1 \pm \sqrt{49}}{4} \\ &= \frac{1 \pm 7}{4} \end{aligned}$$

So, the solutions are $x = \frac{1+7}{4} = 2$ and $x = \frac{1-7}{4} = -\frac{3}{2}$. Quadratic equations can have at most two real solutions, as in the example above. However, some quadratics have only one real solution (e.g., $x^2 + 4x + 4 = 0$; solution: $x = -2$), and some have no real solutions (e.g., $x^2 + x + 5 = 0$).

Some quadratics can be solved more quickly by *factoring*. In the original example,

$$2x^2 - x - 6 = (2x + 3)(x - 2) = 0.$$

Since $(2x + 3)(x - 2) = 0$, either $2x + 3 = 0$ or $x - 2 = 0$ must be true. Therefore,

$$\begin{aligned} 2x + 3 &= 0 & & & x - 2 &= 0 \\ 2x &= -3 & &\text{OR} & x &= 2 \\ x &= -\frac{3}{2} \end{aligned}$$

Other examples of factorable quadratic equations are:

(a)

$$\begin{aligned} x^2 + 8x + 15 &= 0 \\ (x + 3)(x + 5) &= 0 \end{aligned}$$

Therefore, $x + 3 = 0$; $x = -3$

or $x + 5 = 0$; $x = -5$

(b)

$$\begin{aligned} 4x^2 - 9 &= 0 \\ (2x + 3)(2x - 3) &= 0 \end{aligned}$$

$$\text{Therefore,} \quad 2x + 3 = 0; \; x = -\frac{3}{2}$$

$$\text{or} \quad 2x - 3 = 0; \; x = \frac{3}{2}$$

2.6 Inequalities

Any mathematical statement that uses one of the following symbols is called an *inequality*.

$\neq$ "not equal to"
$<$ "less than"
$\leq$ "less than or equal to"
$>$ "greater than"
$\geq$ "greater than or equal to"

For example, the inequality $4x - 1 \leq 7$ states that "$4x - 1$ is less than or equal to 7." To *solve* an inequality means to find the values of the variable that make the inequality true. The approach used to solve an inequality is similar to that used to solve an equation. That is, by using basic operations, you try to isolate the variable on one side of the inequality. The basic rules for solving inequalities are similar to the rules for solving equations, namely:

(i) When the same constant is added to (or subtracted from) both sides of an inequality, the direction of inequality is preserved, and the new inequality is equivalent to the original.
(ii) When both sides of the inequality are multiplied (or divided) by the same constant, the direction of inequality is *preserved if the constant is positive*, but *reversed if the constant is negative*. In either case the new inequality is equivalent to the original.

For example, to solve the inequality $-3x + 5 \leq 17$,

$$-3x + 5 \leq 17$$

$$-3x \leq 12 \quad \text{(5 subtracted from both sides)}$$

$$\frac{-3x}{-3} \geq \frac{12}{-3} \quad \text{(both sides divided by } -3\text{, which reverses the direction of the inequality)}$$

$$x \geq -4$$

Therefore, the solutions to $-3x + 5 \leq 17$ are all real numbers greater than or equal to -4. Another example follows:

$$\frac{4x + 9}{11} > 5$$

$$4x + 9 > 55 \quad \text{(both sides multiplied by 11)}$$

$$4x > 46 \quad \text{(9 subtracted from both sides)}$$

$$x > \frac{46}{4} \quad \text{(both sides divided by 4)}$$

$$x > 11\frac{1}{2}$$

2.7 Applications

Since algebraic techniques allow for the creation and solution of equations and inequalities, algebra has many real-world applications. Below are a few examples. Additional examples are included in the exercises at the end of this section.

Example 1. Ellen has received the following scores on 3 exams: 82, 74, and 90. What score will Ellen need to attain on the next exam so that the average (arithmetic mean) for the 4 exams will be 85 ?

Solution: If x represents the score on the next exam, then the arithmetic mean of 85 will be equal to

$$\frac{82 + 74 + 90 + x}{4}.$$

So,

$$\begin{aligned} \frac{246 + x}{4} &= 85 \\ 246 + x &= 340 \\ x &= 94 \end{aligned}$$

Therefore, Ellen would need to attain a score of 94 on the next exam.

Example 2. A mixture of 12 ounces of vinegar and oil is 40 percent vinegar (by weight). How many ounces of oil must be added to the mixture to produce a new mixture that is only 25 percent vinegar?

Solution: Let x represent the number of ounces of oil to be added. Therefore, the total number of ounces of vinegar in the new mixture will be (0.40)(12), and the total number of ounces of new mixture will be $12 + x$. Since the new mixture must be 25 percent vinegar,

$$\frac{(0.40)(12)}{12 + x} = 0.25.$$

Therefore,

$$\begin{aligned} (0.40)(12) &= (12 + x)(0.25) \\ 4.8 &= 3 + 0.25x \\ 1.8 &= 0.25x \\ 7.2 &= x \end{aligned}$$

Thus, 7.2 ounces of oil must be added to reduce the percent of vinegar in the mixture from 40 percent to 25 percent.

Example 3. In a driving competition, Jeff and Dennis drove the same course at average speeds of 51 miles per hour and 54 miles per hour, respectively. If it took Jeff 40 minutes to drive the course, how long did it take Dennis?

Solution: Let x equal the time, in minutes, that it took Dennis to drive the course. Since distance (d) equals rate (r) multiplied by time (t), i.e.,

$$d = (r)(t),$$

the distance traveled by Jeff can be represented by $(51)\left(\frac{40}{60}\right)$, and the distance traveled by Dennis, $(54)\left(\frac{x}{60}\right)$. Since the distances are equal,

$$\begin{aligned} (51)\left(\frac{40}{60}\right) &= (54)\left(\frac{x}{60}\right) \\ 34 &= 0.9x \\ 37.8 &\approx x \end{aligned}$$

Thus, it took Dennis approximately 37.8 minutes to drive the course. Note: since rates are given in miles per *hour*, it was necessary to express time in hours (i.e., 40 minutes equals $\frac{40}{60}$, or $\frac{2}{3}$, of an hour.)

Example 4. If it takes 3 hours for machine A to produce N identical computer parts, and it takes machine B only 2 hours to do the same job, how long would it take to do the job if both machines worked simultaneously?

Solution: Since machine A takes 3 hours to do the job, machine A can do $\frac{1}{3}$ of the job in 1 hour. Similarly, machine B can do $\frac{1}{2}$ of the job in 1 hour. And if we let x represent the number of hours it would take for the machines working simultaneously to do the job, the two machines would do $\frac{1}{x}$ of the job in 1 hour. Therefore,

$$\begin{aligned} \frac{1}{3} + \frac{1}{2} &= \frac{1}{x} \\ \frac{2}{6} + \frac{3}{6} &= \frac{1}{x} \\ \frac{5}{6} &= \frac{1}{x} \\ \frac{6}{5} &= x \end{aligned}$$

Thus, working together, the machines take only $\frac{6}{5}$ hours, or 1 hour and 12 minutes, to produce the N computer parts.

Example 5. At a fruit stand, apples can be purchased for \$0.15 each and pears for \$0.20 each. At these rates, a bag of apples and pears was purchased for \$3.80. If the bag contained exactly 21 pieces of fruit, how many were pears?

Solution: If a represents the number of apples purchased and p represents the number of pears purchased, two equations can be written as follows:

$$\begin{aligned} 0.15a + 0.20p &= 3.80 \\ a + p &= 21 \end{aligned}$$

From the second equation, $a = 21 - p$. Substituting $21 - p$ into the first equation for a gives

$$\begin{aligned} 0.15(21 - p) + 0.20p &= 3.80 \\ (0.15)(21) - 0.15p + 0.20p &= 3.80 \\ 3.15 - 0.15p + 0.20p &= 3.80 \\ 0.05p &= 0.65 \\ p &= 13 \quad \text{(pears)} \end{aligned}$$

Example 6. It costs a manufacturer \$30 each to produce a particular radio model, and it is assumed that if 500 radios are produced, all will be sold. What must be the selling price per radio to ensure that the *profit* (revenue from sales minus total cost to produce) on the 500 radios is greater than \$8,200 ?

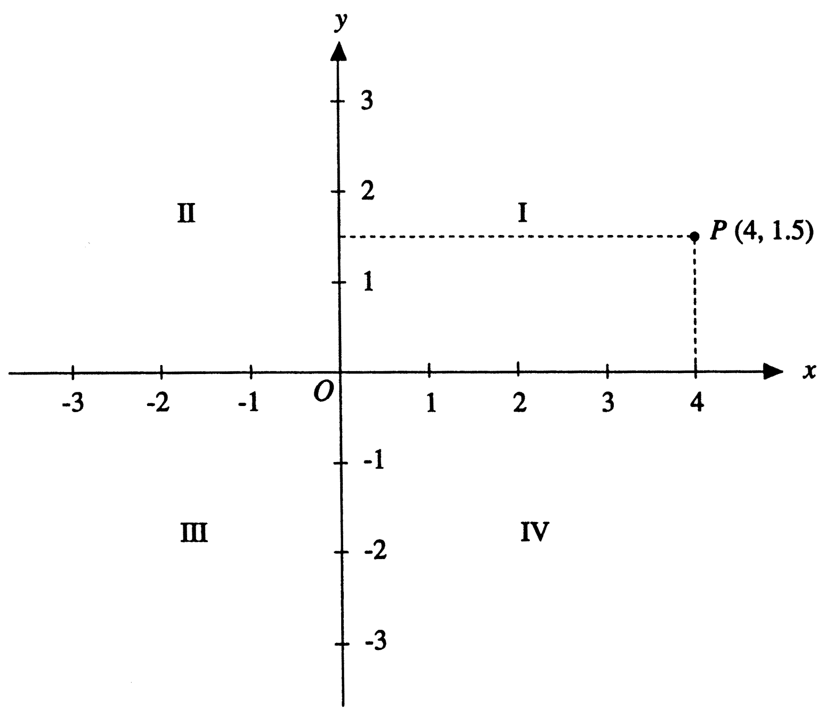
Solution: If y represents the selling price per radio, then the profit must be $500(y - 30)$. Therefore,

$$\begin{aligned} 500(y - 30) &> 8{,}200 \\ 500y - 15{,}000 &> 8{,}200 \\ 500y &> 23{,}200 \\ y &> 46.40 \end{aligned}$$

Thus, the selling price must be greater than \$46.40 to make the profit greater than \$8,200.

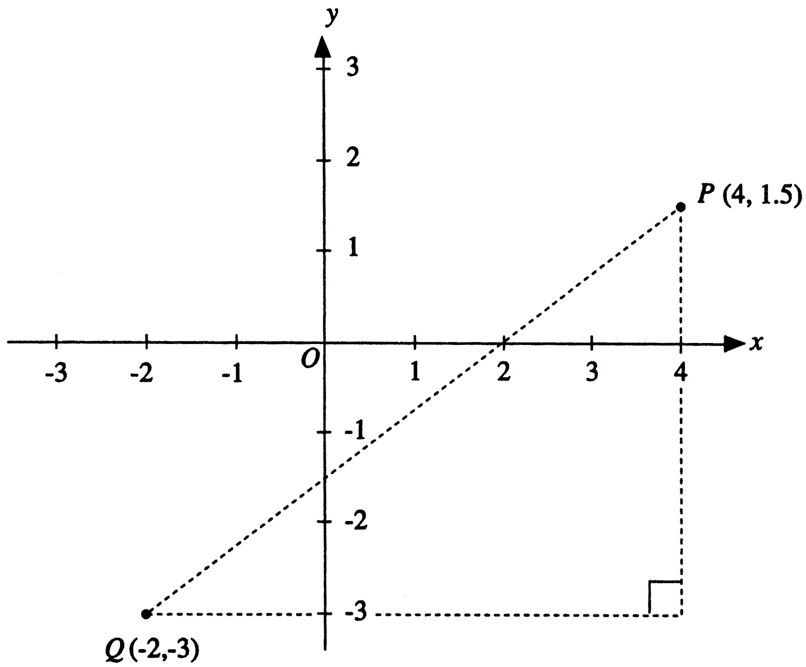
2.8 Coordinate Geometry

Two real number lines (as described in Section 1.5) intersecting at right angles at the zero point on each number line define a *rectangular coordinate system*, often called the *xy-coordinate system* or *xy-plane*. The horizontal number line is called the x-axis, and the vertical number line is called the y-axis. The lines divide the plane into four regions called *quadrants* (I, II, III, and IV) as shown below.



Each point in the system can be identified by an ordered pair of real numbers, (x, y), called *coordinates*. The x-coordinate expresses distance to the left (if negative) or right (if positive) of the y-axis, and the y-coordinate expresses distance below (if negative) or above (if positive) the x-axis. For example, since point P, shown above, is 4 units to the right of the y-axis and 1.5 units above the x-axis, it is identified by the ordered pair (4, 1.5). The *origin* O has coordinates (0, 0). Unless otherwise noted, the units used on the x-axis and the y-axis are the same.

To find the distance between two points, say $P(4, 1.5)$ and $Q(-2, -3)$, represented by the length of line segment PQ in the figure below, first construct a right triangle (see dotted lines) and then note that the two shorter sides of the triangle have lengths 6 and 4.5.



Since the distance between P and Q is the length of the hypotenuse, we can apply the Pythagorean Theorem, as follows:

$$PQ = \sqrt{(6)^2 + (4.5)^2} = \sqrt{56.25} = 7.5$$

(For a discussion of right triangles and the Pythagorean Theorem, see Section 3.3.)

A straight line in a coordinate system is a *graph* of a *linear equation* of the form $y = mx + b$, where m is called the *slope* of the line and b is called the *y-intercept*. The slope of a line passing through points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is defined as

$$\text{slope} = \frac{y_1 - y_2}{x_1 - x_2} \qquad (x_1 \neq x_2).$$

For example, in the coordinate system shown above, the slope of the line passing through points $P(4, 1.5)$ and $Q(-2, -3)$ is

$$\text{slope} = \frac{1.5 - (-3)}{4 - (-2)} = \frac{4.5}{6} = 0.75.$$

The y-intercept is the y-coordinate of the point at which the graph intersects the y-axis. The y-intercept of line PQ in the example above appears to be about -1.5, since line PQ intersects the y-axis close to the point $(0, -1.5)$. This can be confirmed by using the equation of the line, $y = 0.75x + b$, by substituting the coordinates of point Q (or any point that is known to be on the line) into the equation, and by solving for the y-intercept, b, as follows:

$$\begin{aligned} y &= 0.75x + b \\ -3 &= (0.75)(-2) + b \\ b &= -3 + (0.75)(2) \\ b &= -1.5 \end{aligned}$$

The *x-intercept* of the line is the x-coordinate of the point at which the graph intersects the x-axis. One can see from the graph that the x-intercept of line PQ is 2 since PQ passes through the point (2, 0). Also, one can see that the coordinates (2, 0) satisfy the equation of line PQ, which is $y = 0.75x - 1.5$.

ALGEBRA EXERCISES

(Answers on pages 34 and 35)

1. Find an algebraic expression to represent each of the following.

   (a) The square of y is subtracted from 5, and the result is multiplied by 37.
   (b) Three times x is squared, and the result is divided by 7.
   (c) The product of $(x + 4)$ and y is added to 18.

2. Simplify each of the following algebraic expressions by doing the indicated operations, factoring, or combining terms with the same variable part.

   (a) $3x^2 - 6 + x + 11 - x^2 + 5x$
   (b) $3(5x - 1) - x + 4$
   (c) $\dfrac{\left(x^2 + 9\right) - 25}{x - 4}$ $\quad (x \neq 4)$
   (d) $(2x + 5)(3x - 1)$

3. What is the value of the function defined by $f(x) = 3x^2 - 7x + 23$ when $x = -2$?

4. If the function g is defined for all nonzero numbers y by $g(y) = \dfrac{y}{|y|}$, what is the value of $g(2) - g(-200)$?

5. Use the rules of exponents to simplify the following.

   (a) $\left(n^5\right)\left(n^{-3}\right)$
   (b) $\left(s^7\right)\left(t^7\right)$
   (c) $\dfrac{r^{12}}{r^4}$
   (d) $\left(\dfrac{2a}{b}\right)^5$
   (e) $\left(w^5\right)^{-3}$
   (f) $\left(5^0\right)\left(d^3\right)$
   (g) $\dfrac{\left(x^{10}\right)\left(y^{-1}\right)}{\left(x^{-5}\right)\left(y^5\right)}$
   (h) $\left(\dfrac{3x}{y}\right)^2 \div \left(\dfrac{1}{y}\right)^5$

6. Solve each of the following equations for x.

   (a) $5x - 7 = 28$
   (b) $12 - 5x = x + 30$
   (c) $5(x + 2) = 1 - 3x$
   (d) $(x + 6)(2x - 1) = 0$
   (e) $x^2 + 5x - 14 = 0$
   (f) $3x^2 + 10x - 8 = 0$

7. Solve each of the following systems of equations for x and y.

(a) $x + y = 24$
$x - y = 18$

(b) $3x - y = 20$
$x + 2y = 30$

(c) $15x - 18 - 2y = -3x + y$
$10x + 7y + 20 = 4x + 2$

8. Solve each of the following inequalities for x.

(a) $-3x > 7 + x$

(b) $25x + 16 \geq 10 - x$

(c) $16 + x > 8x - 12$

9. Solve for x and y.

$x = 2y$
$5x < y + 7$

10. For a given two-digit positive integer, the tens digit is 5 greater than the units digit. The sum of the digits is 11. Find the integer.

11. If the ratio of $2x$ to $5y$ is 3 to 4, what is the ratio of x to y?

12. Kathleen's weekly salary was increased 8 percent to $237.60. What was her weekly salary before the increase?

13. A theater sells children's tickets for half the adult ticket price. If 5 adult tickets and 8 children's tickets cost a total of $27, what is the cost of an adult ticket?

14. Pat invested a total of $3,000. Part of the money yields 10 percent interest per year, and the rest yields 8 percent interest per year. If the total yearly interest from this investment is $256, how much did Pat invest at 10 percent and how much at 8 percent?

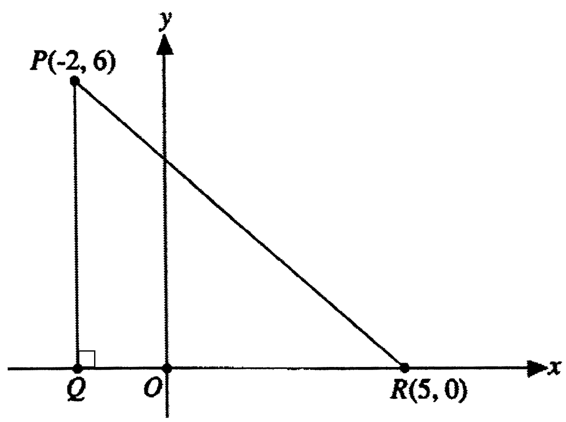
15. Two cars started from the same point and traveled on a straight course in opposite directions for exactly 2 hours, at which time they were 208 miles apart. If one car traveled, on average, 8 miles per hour faster than the other car, what was the average speed for each car for the 2-hour trip?

16. A group can charter a particular aircraft at a fixed total cost. If 36 people charter the aircraft rather than 40 people, the cost per person is greater by $12. What is the cost per person if 40 people charter the aircraft?

17. If 3 times Jane's age, in years, is equal to 8 times Beth's age, in years, and the difference between their ages is 15 years, how old are Jane and Beth?

18. In the coordinate system below, find the

    (a) coordinates of point Q
    (b) perimeter of $\triangle PQR$
    (c) area of $\triangle PQR$
    (d) slope, y-intercept, and equation of the line passing through points P and R



19. In the xy-plane, find the

    (a) slope and y-intercept of a graph with equation $2y + x = 6$
    (b) equation of the straight line passing through the point (3, 2) with y-intercept 1
    (c) y-intercept of a straight line with slope 3 that passes through the point $(-2,\ 1)$
    (d) x-intercepts of the graphs in (a), (b), and (c)

ANSWERS TO ALGEBRA EXERCISES

1. (a) $37(5 - y^2)$, or $185 - 37y^2$

   (b) $\frac{(3x)^2}{7}$, or $\frac{9x^2}{7}$

   (c) $18 + (x + 4)(y)$, or $18 + xy + 4y$

2. (a) $2x^2 + 6x + 5$

   (b) $14x + 1$

   (c) $x + 4$

   (d) $6x^2 + 13x - 5$

3. 49

4. 2

5. (a) n^2

   (b) $(st)^7$

   (c) r^8

   (d) $\frac{32a^5}{b^5}$

   (e) $\frac{1}{w^{15}}$

   (f) d^3

   (g) $\frac{x^{15}}{y^6}$

   (h) $9x^2y^3$

6. (a) 7

   (b) -3

   (c) $-\frac{9}{8}$

   (d) $-6, \frac{1}{2}$

   (e) $-7, 2$

   (f) $\frac{2}{3}, -4$

7. (a) $x = 21$
   $y = 3$

   (b) $x = 10$
   $y = 10$

   (c) $x = \frac{1}{2}$
   $y = -3$

8. (a) $x < -\frac{7}{4}$

   (b) $x \geq -\frac{3}{13}$

   (c) $x < 4$

9. $x < \frac{14}{9}$, $y < \frac{7}{9}$

10. 83

11. 15 to 8

12. \$220

13. \$3

14. \$800 at 10%; \$2,200 at 8%

15. 48 mph and 56 mph

16. \$108

17. Beth is 9; Jane is 24.

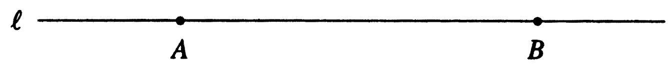
18. (a) $(-2,\ 0)$
    (b) $13 + \sqrt{85}$
    (c) 21
    (d) slope = $\frac{-6}{7}$, y-intercept = $\frac{30}{7}$, $y = \frac{-6}{7}x + \frac{30}{7}$, or $7y + 6x = 30$

19. (a) slope = $-\frac{1}{2}$, y-intercept = 3
    (b) $y = \frac{x}{3} + 1$
    (c) 7
    (d) $6,\ -3,\ -\frac{7}{3}$

GEOMETRY

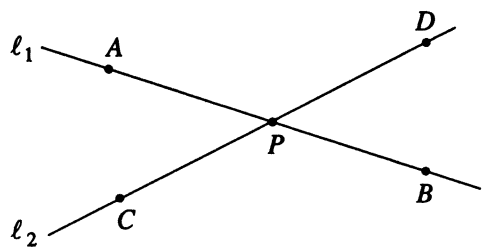
3.1 Lines and Angles

In geometry, a basic building block is the *line*, which is understood to be a "straight" line. It is also understood that lines are *infinite* in length. In the figure below, A and B are points on line ℓ.



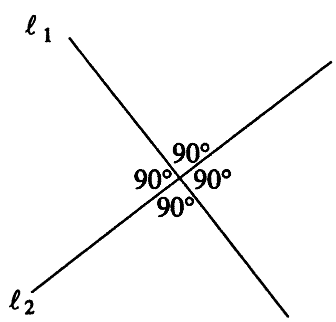
That part of line ℓ from A to B, including the endpoints A and B, is called a *line segment*, which is *finite* in length. Sometimes the notation "AB" denotes line segment AB and sometimes it denotes the *length* of line segment AB. The exact meaning of the notation can be determined from the context.

Lines ℓ_1 and ℓ_2, shown below, intersect at point P. Whenever two lines intersect at a single point, they form four angles.

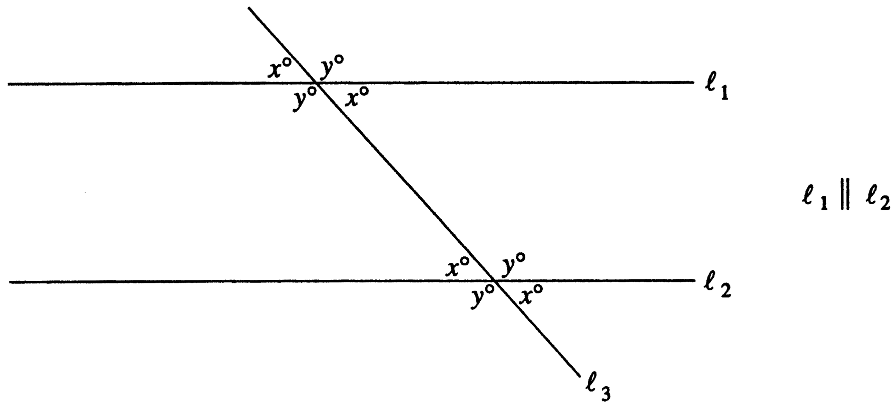


Opposite angles, called *vertical angles*, are the same size, i.e., have *equal measure*. Thus, $\angle APC$ and $\angle DPB$ have equal measure, and $\angle APD$ and $\angle CPB$ also have equal measure. The sum of the measures of the four angles is $360°$.

If two lines, ℓ_1 and ℓ_2, intersect such that all four angles have equal measure (see figure below), we say that the lines are *perpendicular*, or $\ell_1 \perp \ell_2$, and each of the four angles has a measure of $90°$. An angle that measures $90°$ is called a *right angle*, and an angle that measures $180°$ is called a *straight angle*.

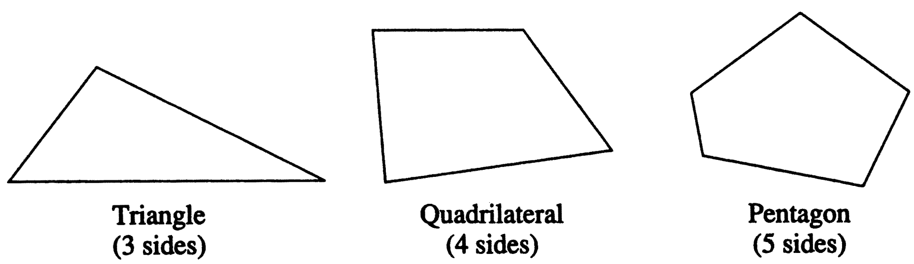


If two distinct lines in the same plane do not intersect, the lines are said to be *parallel*. The figure below shows two parallel lines, ℓ_1 and ℓ_2, which are intersected by a third line, ℓ_3, forming eight angles. Note that four of the angles have equal measure ($x°$) and the remaining four have equal measure ($y°$) where $x + y = 180$.



3.2 Polygons

A *polygon* is a closed figure formed by the intersection of three or more line segments, called *sides*, with all intersections at endpoints, called *vertices*. In this discussion, the term "polygon" will mean "convex polygon," that is, a polygon in which the measure of each interior angle is less than 180°. The figures below are examples of such polygons.



The sum of the measures of the interior angles of an n-sided polygon is $(n - 2)(180°)$. For example, the sum for a triangle ($n = 3$) is $(3 - 2)(180°) = 180°$, and the sum for a *hexagon* ($n = 6$) is $(6 - 2)(180°) = 720°$.

A polygon with all sides the same length and the measures of all interior angles equal is called a *regular polygon*. For example, in a *regular octagon* (8 sides of equal length), the sum of the measures of the interior angles is $(8 - 2)(180°) = 1{,}080°$. Therefore, the measure of each angle is $1{,}080° \div 8 = 135°$.

The *perimeter* of a polygon is defined as the sum of the lengths of its sides. The *area* of a polygon is the measure of the area of the region enclosed by the polygon.

In the next two sections, we look at some basic properties of the simplest polygons—triangles and quadrilaterals.

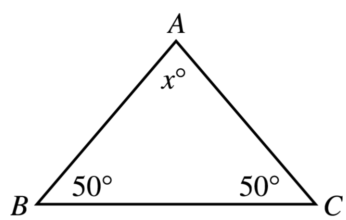
3.3 Triangles

Every triangle has three sides and three interior angles whose measures sum to $180°$. It is also important to note that the length of each side must be shorter than the sum of the lengths of the other two sides. For example, the sides of a triangle could not have lengths of 4, 7, and 12 because 12 is not shorter than $4 + 7$.

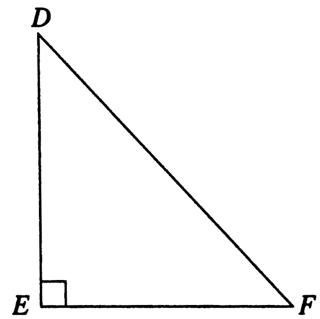
The following are special triangles.

(a) A triangle with all sides of equal length is called an *equilateral triangle*. The measures of three interior angles of such a triangle are also equal (each 60°).

(b) A triangle with at least two sides of equal length is called an *isosceles triangle*. If a triangle has two sides of equal length, then the measures of the angles opposite the two sides are equal. The converse of the previous statement is also true. For example, in $\triangle ABC$ below, since both $\angle ABC$ and $\angle BCA$ have measure $50°$, it must be true that $BA = AC$. Also, since $50 + 50 + x = 180$, the measure of $\angle BAC$ must be $80°$.



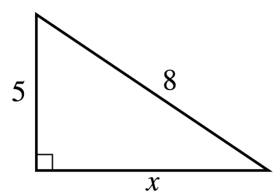
(c) A triangle with an interior angle that has measure 90° is called a *right triangle*. The two sides that form the 90° angle are called *legs* and the side opposite the 90° angle is called the *hypotenuse*.



For right $\triangle DEF$ above, DE and EF are legs and DF is the hypotenuse. The *Pythagorean Theorem* states that for any right triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the legs. Thus, in right $\triangle DEF$

$$(DF)^2 = (DE)^2 + (EF)^2$$

This relationship can be used to find the length of one side of a right triangle if the lengths of the other two sides are known. For example, if one leg of a right triangle has length 5 and the hypotenuse has length 8, then the length of the other side can be calculated as follows:

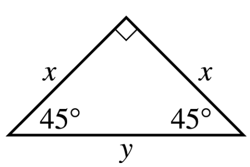


$$8^2 = 5^2 + x^2$$
$$64 = 25 + x^2$$
$$39 = x^2$$

Since $x^2 = 39$ and x must be positive, $x = \sqrt{39}$, or approximately 6.2.

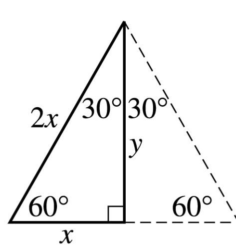
The Pythagorean Theorem can be used to determine the ratios of the sides of two special right triangles:

An isosceles right triangle has angles measuring 45°, 45°, 90°. The Pythagorean Theorem applied to the triangle below shows that the lengths of its sides are in the ratio 1 to 1 to $\sqrt{2}$.



$$y^2 = x^2 + x^2$$
$$y^2 = 2x^2$$
$$y = \sqrt{2}x$$

A $30° - 60° - 90°$ right triangle is half of an equilateral triangle, as the following figure shows.



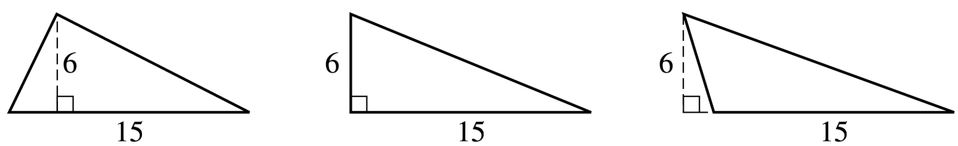
So the length of the shortest side is half the longest side, and by the Pythagorean Theorem, the ratio of all three side lengths is 1 to $\sqrt{3}$ to 2, since

$$\begin{aligned} x^2 + y^2 &= (2x)^2 \\ x^2 + y^2 &= 4x^2 \\ y^2 &= 4x^2 - x^2 \\ y^2 &= 3x^2 \\ y &= \sqrt{3}x \end{aligned}$$

The *area* of a triangle is defined as half the length of a base (b) multiplied by the corresponding height (h), that is,

$$\text{Area} = \frac{bh}{2}.$$

Any side of a triangle may be considered a base, and then the corresponding height is the perpendicular distance from the opposite vertex to the base (or an extension of the base). The examples below summarize three possible locations for measuring height with respect to a base.

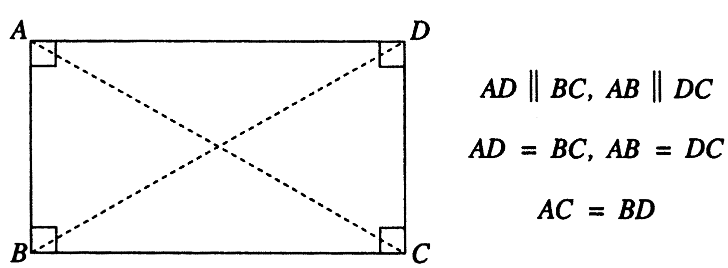


In all three triangles above, the area is $\frac{(15)(6)}{2}$, or 45.

3.4 Quadrilaterals

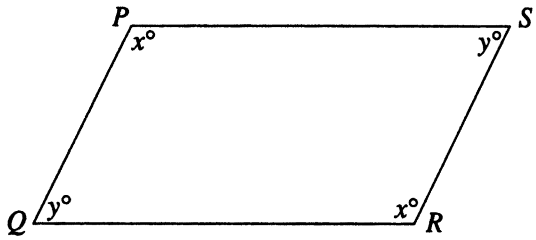
Every quadrilateral has four sides and four interior angles whose measures sum to 360° . The following are special quadrilaterals.

(a) A quadrilateral with all interior angles of equal measure (each 90°) is called a *rectangle*. Opposite sides are parallel and have equal length, and the two diagonals have equal length.



A rectangle with all sides of equal length is called a *square*.

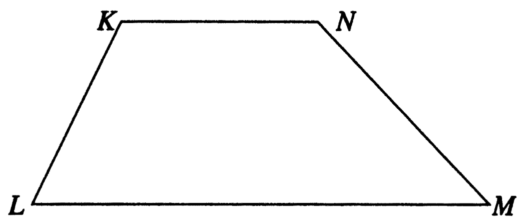
(b) A quadrilateral with both pairs of opposite sides parallel is called a *parallelogram*. In a parallelogram, opposite sides have equal length, and opposite interior angles have equal measure.



$PS \parallel QR,\ PQ \parallel SR$

$PS = QR,\ PQ = SR$

(c) A quadrilateral with one pair of opposite sides parallel is called a *trapezoid*.

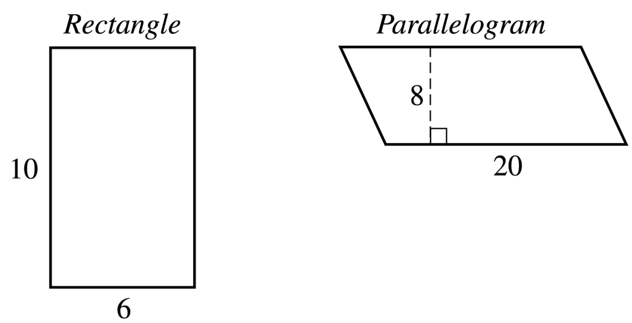


$KN \parallel LM$

For all rectangles and parallelograms the *area* is defined as the length of the base (b) multiplied by the height (h), that is

$$\text{Area} = bh$$

Any side may be considered a base, and then the height is either the length of an adjacent side (for a rectangle) or the length of a perpendicular line from the base to the opposite side (for a parallelogram). Here are examples of each:



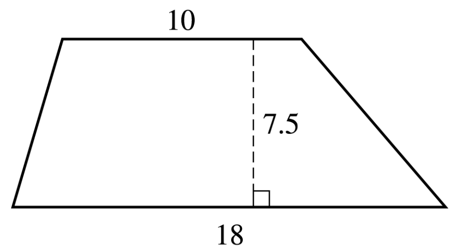
Area = (6)(10) = 60 Area = (20)(8) = 160

The area of a trapezoid may be calculated by finding half the sum of the lengths of the two parallel sides $(b_1 \text{ and } b_2)$ and then multiplying the result by the height (h), that is,

$$\text{Area} = \frac{1}{2}(b_1 + b_2)(h).$$

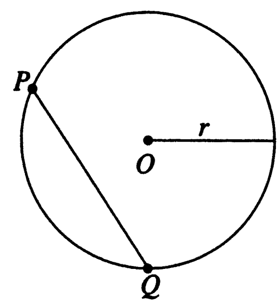
For example, for the trapezoid shown below with bases of length 10 and 18, and a height of 7.5,

$$\text{Area} = \frac{1}{2}(10 + 18)(7.5) = 105.$$



3.5 Circles

The set of all points in a plane that are a given distance r from a fixed point O is called a *circle*. The point O is called the *center* of the circle, and the distance r is called *the radius* of the circle. Also, any line segment connecting point O to a point on the circle is called *a radius*.



Any line segment that has its endpoints on a circle, such as PQ above, is called *a chord*. Any chord that passes through the center of a circle is called *a diameter*. The length of a diameter is called *the diameter* of a circle. Therefore, the diameter of a circle is always equal to twice its radius.

The distance around a circle is called its *circumference* (comparable to the perimeter of a polygon). In any circle, the ratio of the circumference c to the diameter d is a fixed constant, denoted by the Greek letter π:

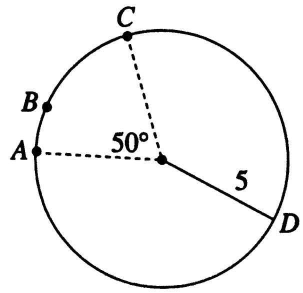
$$\frac{c}{d} = \pi$$

The value of π is approximately 3.14 and may also be approximated by the fraction $\frac{22}{7}$. If r is the radius of the circle, then $\frac{c}{2r} = \pi$, so the circumference is related to the radius by the equation

$$c = 2\pi r.$$

Therefore, if a circle has a radius equal to 5.2, then its circumference is $(2)(\pi)(5.2) = (10.4)(\pi)$, which is approximately equal to 32.7.

On a circle, the set of all points between and including two given points is called an *arc*. It is customary to refer to an arc with three points to avoid ambiguity. In the figure below, arc ABC is the short arc from A to C, but arc ADC is the long arc from A to C in the reverse direction.



Arcs can be measured in degrees. The number of degrees of arc equals the number of degrees in the central angle formed by the two radii intersecting the arc's endpoints. The number of degrees of arc in the entire circle (one complete revolution) is 360. Thus, in the figure above, arc ABC is a $50°$ arc and arc ADC is a $310°$ arc.

To find the *length* of an arc, it is important to know that the ratio of arc length to circumference is equal to the ratio of arc measure (in degrees) to 360. In the figure above, the circumference is 10π. Therefore,

$$\frac{\text{length of arc } ABC}{10\pi} = \frac{50}{360}$$

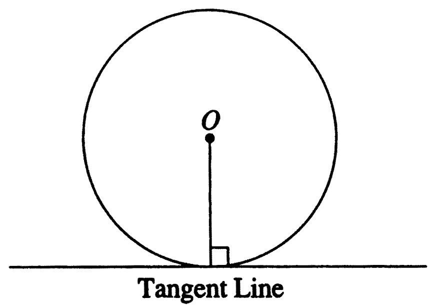
$$\text{length of arc } ABC = \left(\frac{50}{360}\right)(10\pi) = \frac{25\pi}{18}$$

The *area* of a circle with radius r is equal to πr^2. For example, the area of the circle above is $\pi(5)^2 = 25\pi$. In this circle, the pie-shaped region bordered by arc ABC and the two dashed radii is called a *sector* of the circle, with central angle $50°$. Just as in the case of arc length, the ratio of the area of the sector to the area of the entire circle is equal to the ratio of the arc measure (in degrees) to 360. So if S represents the area of the sector with central angle $50°$, then

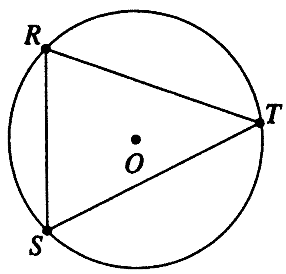
$$\frac{S}{25\pi} = \frac{50}{360}$$

$$S = \left(\frac{50}{360}\right)(25\pi) = \frac{125\pi}{36}$$

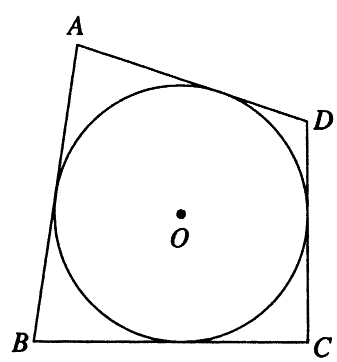
A *tangent* to a circle is a line that has exactly one point in common with the circle. A radius with its endpoint at the point of tangency is perpendicular to the tangent line. The converse is also true.



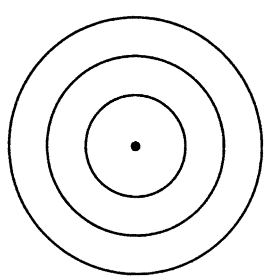
If each vertex of a polygon lies on a circle, then the polygon is *inscribed* in the circle, or equivalently, the circle is *circumscribed* about the polygon. Triangle *RST* below is inscribed in the circle with center *O*.



If each side of a polygon is tangent to a given circle, then the polygon is *circumscribed* about the circle, or equivalently, the circle is *inscribed* in the polygon. In the figure below, quadrilateral *ABCD* is circumscribed about the circle with center *O*.



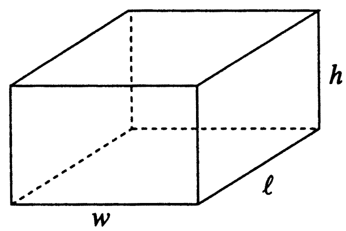
Two or more circles with the same center are called *concentric* circles.



3.6 Three-Dimensional Figures

Basic three-dimensional figures include rectangular solids, cubes, cylinders, spheres, pyramids, and cones. In this section, we look at some properties of rectangular solids and right circular cylinders.

(a) A *rectangular solid* has six rectangular surfaces called *faces* (see figure below). Each line segment shown is called an *edge* (there are 12 edges), and each point at which the edges meet is called a *vertex* (there are 8 vertices). The dimensions of a rectangular solid are length (ℓ), width (w), and height (h).



A rectangular solid with $\ell = w = h$ is called a *cube*. The *volume* V of a rectangular solid is the product of the three dimensions,

$$V = \ell wh$$

The *surface area* A of a rectangular solid is the sum of the areas of the six faces, or

$$A = 2(w\ell + \ell h + wh).$$

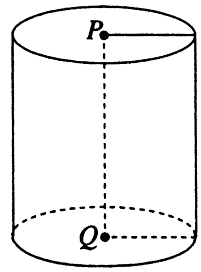
For example, if a rectangular solid has length 8.5, width 5, and height 10, then its volume is

$$V = (8.5)(5)(10) = 425,$$

and its surface area is

$$A = 2[(5)(8.5) + (8.5)(10) + (5)(10)] = 355.$$

(b) A *right circular cylinder* is shown in the figure below. Its bases are circles with equal radii and centers P and Q, respectively, and its height PQ is perpendicular to both bases.



The *volume* V of a right circular cylinder with a base radius r and height h is the area of the base multiplied by the height, or

$$V = \pi r^2 h.$$

The *surface area* A of a right circular cylinder is the sum of the two base areas and the area of the curved surface, or

$$A = 2\left(\pi r^2\right) + 2\pi rh.$$

For example, if a right circular cylinder has a base radius of 3 and a height of 6.5, then its volume is

$$V = \pi(3)^2(6.5) = 58.5\pi,$$

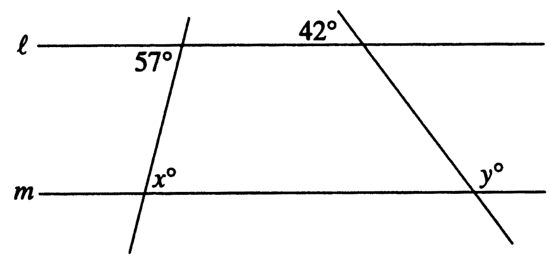
and its surface area is

$$A = (2)(\pi)(3)^2 + (2)(\pi)(3)(6.5) = 57\pi.$$

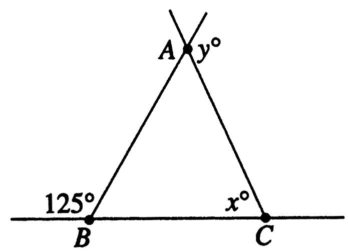
GEOMETRY EXERCISES

(Answers on page 50)

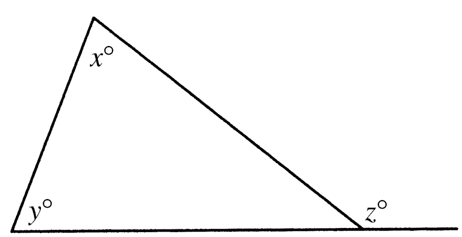
1. Lines ℓ and m below are parallel. Find the values of x and y.



2. In the figure below, $AC = BC$. Find the values of x and y.



3. In the figure below, what relationship must hold among angle measures x, y, and z ?

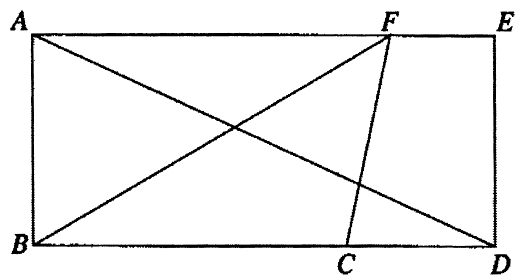


4. What is the sum of the measures of the interior angles of a decagon (10-sided polygon) ?

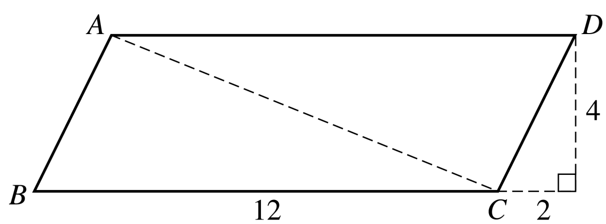
5. If the polygon in #4 is regular, what is the measure of each interior angle?

6. The lengths of two sides of an isosceles triangle are 15 and 22, respectively. What are the possible values of the perimeter?

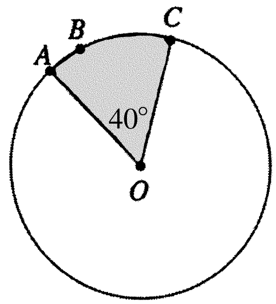
7. In rectangle $ABDE$ below, $AB = 5$, $BC = 7$, and $CD = 3$. Find the
   (a) area of $ABDE$
   (b) area of triangle BCF
   (c) length of AD
   (d) perimeter of $ABDE$



8. In parallelogram $ABCD$ below, find the
   (a) area of $ABCD$
   (b) perimeter of $ABCD$
   (c) length of diagonal AC



9. The circle with center O below has radius 4. Find the
   (a) circumference
   (b) length of arc ABC
   (c) area of the shaded region

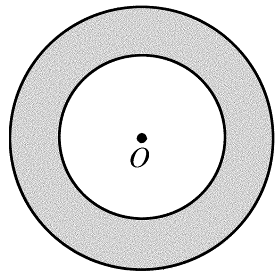


10. The figure below shows two concentric circles each with center O. If the larger circle has radius 12 and the smaller circle has radius 8, find the

(a) circumference of the larger circle

(b) area of the smaller circle

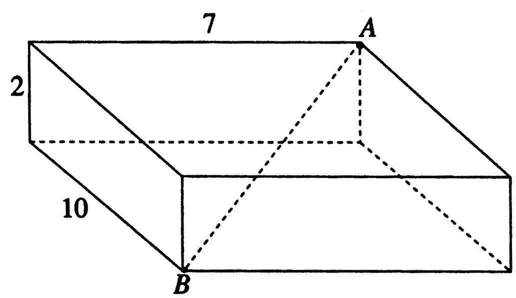
(c) area of the shaded region



11. For the rectangular solid below, find the

(a) surface area

(b) length of diagonal AB



ANSWERS TO GEOMETRY EXERCISES

1. $x = 57, \; y = 138$

2. $x = 70, \; y = 125$

3. $z = x + y$

4. 1,440°

5. 144°

6. 52 or 59

7. (a) 50
   (b) 17.5
   (c) $5\sqrt{5}$
   (d) 30

8. (a) 48
   (b) $24 + 4\sqrt{5}$
   (c) $2\sqrt{29}$

9. (a) 8π
   (b) $\frac{8\pi}{9}$
   (c) $\frac{16\pi}{9}$

10. (a) 24π
    (b) 64π
    (c) 80π

11. (a) 208
    (b) $3\sqrt{17}$

DATA ANALYSIS

4.1 Measures of Central Location

Two common measures of central location, often called “average,” for a discrete set of numerical values or measurements are the *arithmetic mean* and the *median*.

The *average* (*arithmetic mean*) of n values is defined as the *sum of the* n *values divided by* n. For example, the arithmetic mean of the values 5, 8, 8, 14, 15, and 10 is $60 \div 6 = 10$.

If we order the n values from least to greatest, the *median* is defined as *the middle value if* n *is odd* and *the sum of the two middle values divided by 2 if* n *is even*. In the example above, $n = 6$, which is even. Ordered from least to greatest, the values are 5, 8, 8, 10, 14, and 15. Therefore, the median is

$$\frac{8 + 10}{2} = 9.$$

Note that for the same set of values, the arithmetic mean and the median need not be equal, although they could be. For example, the set of values 10, 20, 30, 40, and 50 has arithmetic mean = median = 30.

Another measure of central location is called the *mode*, which is defined as *the most frequently occurring value*. For the six measurements above, the mode is 8.

4.2 Measures of Dispersion

Measures of dispersion, or spread, for a discrete set of numerical values or measurements take many forms in data analyses. The simplest measure of dispersion is called the *range*, which is defined as the *greatest measurement minus the least measurement*. So, in the example in 4.1 above, the range for the six values is 15 minus 5, or 10.

Since the range is affected by only the two most extreme values in the set of measurements, other measures of dispersion have been developed that are affected by every measurement. The most commonly used of these other measures is called the *standard deviation*. The value of the standard deviation for a set of n measurements can be calculated by (1) first calculating the arithmetic mean, (2) finding the difference between that mean and each measurement, (3) squaring each of the differences, (4) summing the squared values, (5) dividing the sum by n, and finally (6) taking the nonnegative square root of the quotient. The following demonstrates this calculation for the example used in 4.1.

x	$x-10$	$(x-10)^2$
5	-5	25
8	-2	4
8	-2	4
10	0	0
14	4	16
15	5	25
		74

$$\text{standard deviation} = \sqrt{\frac{74}{6}} \approx 3.5$$

The standard deviation can be roughly interpreted as the average distance from the arithmetic mean for the n measurements. The standard deviation cannot be negative, and when two sets of measurements are compared, the one with the larger dispersion will have the larger standard deviation.

4.3 Frequency Distributions

For some sets of measurements, it is more convenient and informative to display the measurements in a *frequency distribution*. For example, the following values could represent the number of dependent children in each of 25 families living on a particular street.

1, 2, 0, 4, 1, 3, 3, 1, 2, 0, 4, 5, 2,
3, 2, 3, 2, 4, 1, 2, 3, 0, 2, 3, 1

These data can be grouped into a *frequency distribution* by listing each different value (x) and the frequency (f) of occurrence for each value.

Frequency Distribution

x	f
0	3
1	5
2	7
3	6
4	3
5	1
Total	25

The frequency distribution format not only provides a quick summary of the data, but it also simplifies the calculations of the central location and dispersion measures. For these data, the x's can be summed by multiplying each x by its frequency and then adding the products. So, the arithmetic mean is

$$\frac{(0)(3) + (1)(5) + (2)(7) + (3)(6) + (4)(3) + (5)(1)}{25} = 2.16$$

The median is the middle (13th) x value in order of size. The f values show that the 13th x value must be a 2. The range is 5 minus 0, or 5. The standard deviation can also be calculated more easily from a frequency distribution, although in practice it is likely that a programmable calculator would be used to calculate both the mean and the standard deviation directly from the 25 measurements.

4.4 Counting

Some definitions and principles basic to counting are:

(a) *If one task has* n *possible outcomes and a second task has* m *possible outcomes*, *then the joint occurrence of the two tasks has* (n)(m) *possible outcomes.* For example, if Town A and Town B are joined by 3 different roads, and Town B and Town C are joined by 4 different roads, then the number of different routes from Town A to Town C through B is (3)(4), or 12. Each time a coin is flipped, there are 2 possible outcomes: heads or tails. Therefore, if a coin is flipped 4 times, then the number of possible outcomes is (2)(2)(2)(2), or 16.

(b) *For any integer* n *greater than* 1, *the symbol* n!, *pronounced* "n *factorial,*" *is defined as the product of all positive integers less than or equal to* n. *Also*, 0! = 1! = 1. Therefore,

$$0! = 1$$
$$1! = 1$$
$$2! = (2)(1) = 2$$
$$3! = (3)(2)(1) = 6$$
$$4! = (4)(3)(2)(1) = 24$$

and so on.

(c) *The number of ways that* n *objects can be ordered is* n!. For example, the number of ways that the letters A, B, and C can be ordered is 3!, or 6. The six orders are

ABC, *ACB*, *BAC*, *BCA*, *CAB*, and *CBA*

(d) *The number different subsets of* r *objects that can be selected from* n *objects* (r ≤ n), *without regard to the order of selection, is*

$$\frac{n!}{(n-r)!r!}.$$

For example, the number of different committees of 3 people that can be selected from 5 people is

$$\frac{5!}{(5-3)!3!} = \frac{5!}{2!\,3!} = \frac{120}{(2)(6)} = 10.$$

These 10 subsets are called *combinations* of 5 objects selected 3 at a time.

4.5 Probability

Everyday there are occasions in which decisions must be made in the face of uncertainty. The decision-making process often involves the selection of a course of action based on an analysis of possible outcomes. For situations in which the possible outcomes are all equally likely, the *probability that an event* E *occurs*, represented by "$P(E)$", can be defined as

$$P(E) = \frac{\text{The number of outcomes involving the occurrence of } E}{\text{The total number of possible outcomes}}.$$

For example, if a committee of 11 students consists of 2 seniors, 5 juniors, and 4 sophomores, and one student is to be selected at random to chair the committee, then the probability that the student selected will be a senior is $\frac{2}{11}$.

In general, "$P(E)$" can be thought of as a number assigned to an event E which expresses the likelihood that E occurs. If E cannot occur, then $P(E) = 0$, and if E must occur, then $P(E) = 1$. If the occurrence of E is uncertain, then $0 < P(E) < 1$. The probability that event E does NOT occur is $1 - P(E)$. For example, if the probability is 0.75 that it will rain tomorrow, then the probability that it will not rain tomorrow is $1 - 0.75$, or 0.25.

The probability that events E and F both occur can be represented by $P(E \text{ and } F)$*, and the probability that at least one of the two events occurs can be represented by $P(E \text{ or } F)$**. One of the fundamental relationships among probabilities is called the *Addition Law*:

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F).$$

---

*Many texts use $P(E \cap F)$.

**Many texts use $P(E \cup F)$.

For example, if a card is to be selected randomly from a standard deck of 52 playing cards, E is the event that a heart is selected, and F is the event that a 9 is selected, then

$$P(E) = \frac{13}{52},\ P(F) = \frac{4}{52},\ \text{and}\ P(E \text{ and } F) = \frac{1}{52}.$$

Therefore, $P(E \text{ or } F) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$.

Two events are said to be *independent* if the occurrence or nonoccurrence of either one in no way affects the occurrence of the other. It follows that if events E and F are independent events, then $P(E \text{ and } F) = P(E) \cdot P(F)$. Two events are said to be *mutually exclusive* if the occurrence of either one precludes the occurrence of the other. In other words, if events E and F are mutually exclusive, then $P(E \text{ and } F) = 0$.

Example: If $P(A) = 0.45$ and $P(B) = 0.20$, and the two events are independent, what is $P(A \text{ or } B)$?

According to the Addition Law:

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= P(A) + P(B) - P(A) \cdot P(B) \\ &= 0.45 + 0.20 - (0.45)(0.20) \\ &= 0.56 \end{aligned}$$

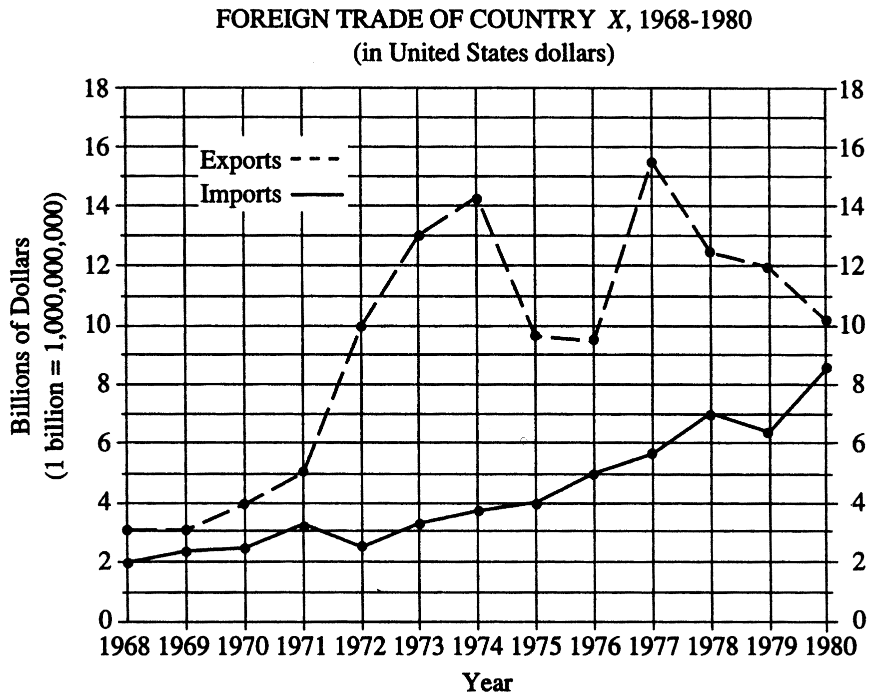
If the two events in the example above had been mutually exclusive, then $P(A \text{ or } B)$ would have been found as follows:

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= 0.45 + 0.20 - 0 \\ &= 0.65 \end{aligned}$$

4.6 Data Representation and Interpretation

Data can be summarized and represented in various forms, including tables, bar graphs, circle graphs, line graphs, and other diagrams. The following are several examples of tables and graphs, each with questions that can be answered by selecting the appropriate information and applying mathematical techniques.

Example 1.



(a) For which year shown on the graph did exports exceed the previous year's exports by the greatest dollar amount?

(b) In 1973 the dollar value of imports was approximately what percent of the dollar value of exports?

(c) If it were discovered that the import dollar amount shown for 1978 was incorrect and should have been \$3.1 billion instead, then the average (arithmetic mean) import dollar amount per year for the 13 years would be how much less?

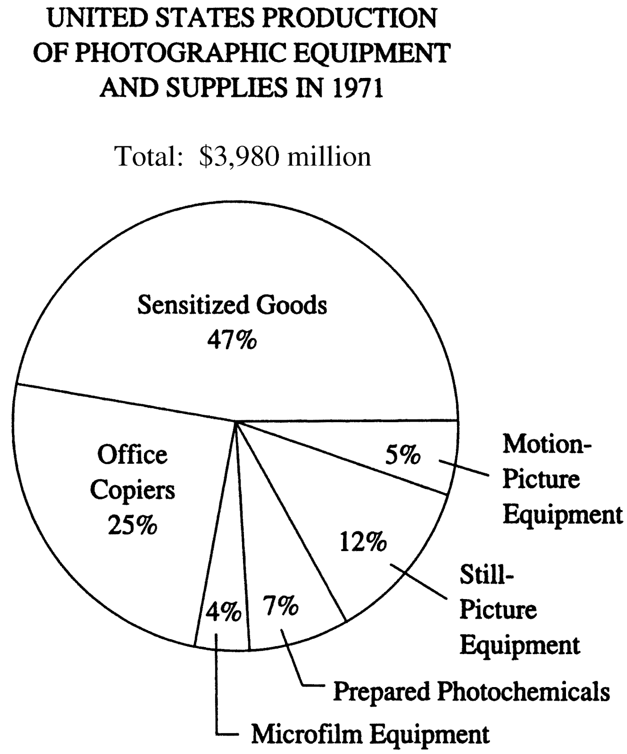
Solutions:

(a) The greatest increase in exports form one year to the next is represented by the dotted line segment with the steepest positive slope, which is found between 1976 and 1977. The increase was approximately \$6 billion. Thus, the answer is 1977.

(b) In 1973, the dollar value of imports was approximately \$3.3 billion, and the dollar value of exports was \$13 billion. Therefore, the answer is $\frac{3.3}{13}$, or approximately 25%.

(c) If the import dollar amount in 1978 were \$3.1 billion, rather than the table amount, \$7 billion, then the sum of the import amounts for the 13 years would be reduced by \$3.9 billion. Therefore, the average per year would be reduced by $\frac{\$3.9}{13}$ billion, which is \$0.3 billion, or \$300 million.

Example 2.



(a) In 1971 what was the ratio of the value of sensitized goods to the value of still-picture equipment produced in the United States?

(b) If the value of office copiers produced in 1971 was 30 percent higher than the corresponding value in 1970, what was the value of office copiers produced in 1970 ?

(c) If the areas of the sectors in the circle graph are drawn in proportion to the percents shown, what is the measure, in degrees, of the central angle of the sector representing the percent of prepared photochemicals produced?

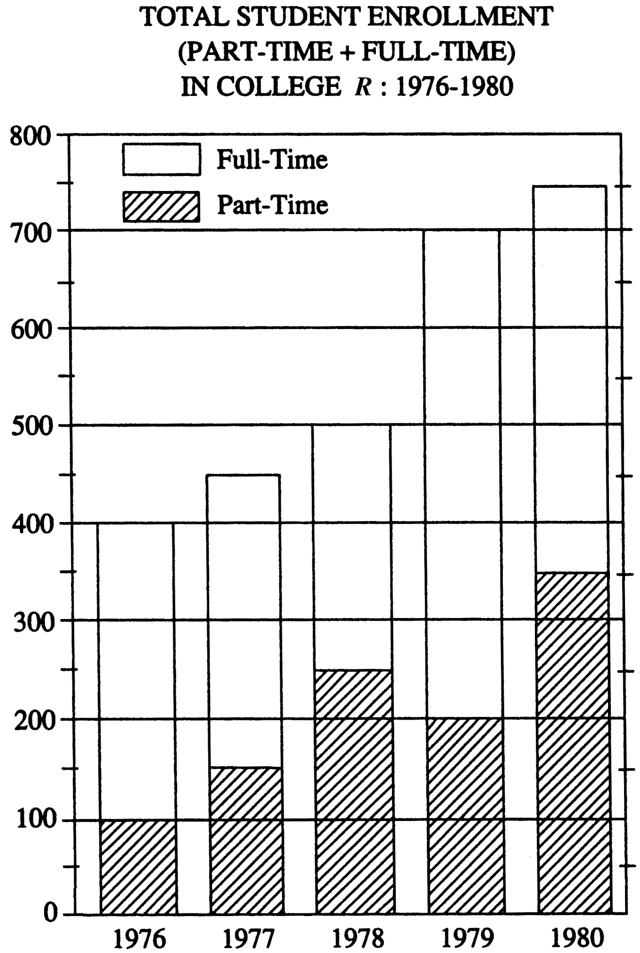
Solutions:

(a) The ratio of the value of sensitized goods to the value of still-picture equipment is equal to the ratio of the corresponding percents shown. Therefore, the ratio is 47 to 12, or approximately 4 to 1.

(b) The value of office copiers produced in 1971 was (0.25)($3,980), or $995 million. Therefore, if the corresponding value in 1970 was x, then $x(1.30)$ = $995 million, or x = $765 million.

(c) Since the sum of the central angles for the six sectors is 360°, the central angle for the sector representing prepared photochemicals is (0.07)(360)°, or 25.2°.

Example 3.



(a) For which year was the ratio of part-time enrollment to total enrollment the greatest?

(b) What was the full-time enrollment in 1977 ?

(c) What was the percent increase in total enrollment from 1976 to 1980 ?

Solutions:

(a) It is visually apparent that the height of the shaded bar compared to the total height of the bar is greatest in 1978 (about half the total height). No calculations are necessary.

(b) In 1977 the total enrollment was approximately 450 students, and the part-time enrollment was approximately 150 students. Thus, the full-time enrollment was $450 - 150$, or 300 students.

(c) The total enrollments for 1976 and 1980 were approximately 400 and 750, respectively. Therefore, the percent increase from 1976 to 1980 was

$$\frac{750 - 400}{400} = \frac{350}{400} = 0.875 = 87.5\%.$$

Example 4.

CONSUMER COMPLAINTS RECEIVED
BY THE CIVIL AERONAUTICS BOARD

Category	1980 (percent)	1981 (percent)
Flight Problems	20.0%	22.1%
Baggage	18.3	21.8
Customer service	13.1	11.3
Oversales of seats	10.5	11.8
Refund problems	10.1	8.1
Fares	6.4	6.0
Reservations and ticketing	5.8	5.6
Tours	3.3	2.3
Smoking	3.2	2.9
Advertising	1.2	1.1
Credit	1.0	0.8
Special passengers	0.9	0.9
Other	6.2	5.3
	100.0%	100.0%
Total Number of Complaints	22,998	13,278

(a) Approximately how many complaints concerning credit were received by the Civil Aeronautics Board in 1980 ?

(b) By approximately what percent did the total number of complaints decrease from 1980 to 1981 ?

(c) Which of the following statements can be inferred from the table?

I. In 1980 and in 1981, complaints about flight problems, baggage, and customer service together accounted for more than 50 percent of all consumer complaints received by the Civil Aeronautics Board.

II. The number of special passenger complaints was unchanged from 1980 to 1981.

III. From 1980 to 1981, the number of flight problem complaints increased by more than 2 percent.

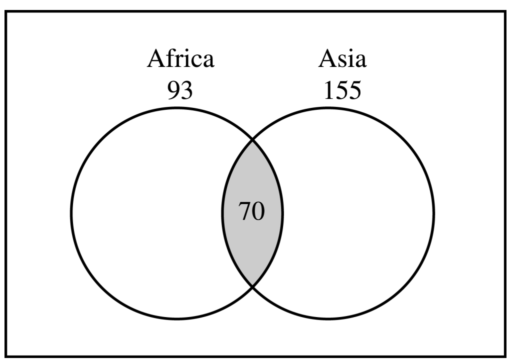
Solutions:

(a) In 1980, 1 percent of the complaints concerned credit, so the number of complaints was approximately (0.01)(22,998), or 230.

(b) The decrease in total complaints from 1980 to 1981 was $22{,}998 - 13{,}278$, or 9,720. Therefore, the percent decrease was $9{,}720 \div 22{,}998$, or 42 percent.

(c) Since $20.0 + 18.3 + 13.1$ and $22.1 + 21.8 + 11.3$ are both greater than 50, statement I is true. The percent of special passenger complaints did remain the same for 1980 to 1981, but the *number* of special passenger complaints decreased because the total number of complaints decreased. Thus, statement II is false. The percents shown in the table for flight problems do in fact increase more than 2 percentage points. However, the *number* of flight problem complaints in 1980 was (0.2)(22,998), or 4,600, and the number in 1981 was (0.221)(13,278), or 2,934. So, the number of flight problem complaints actually decreased from 1980 to 1981. Therefore, statement I is the only statement that can be inferred from the table.

Example 5.



In a survey of 250 European travelers, 93 have traveled to Africa, 155 have traveled to Asia, and 70 have traveled to both of these continents, as illustrated in the *Venn diagram* above.

(a) How many of the travelers surveyed have traveled to Africa but not to Asia?

(b) How many of the travelers surveyed have traveled to at least one of the two continents Africa and Asia?

(c) How many of the travelers surveyed have traveled neither to Africa nor to Asia?

Solutions:

A Venn diagram is useful for sorting out various sets and subsets that may overlap. The rectangular region represents the set of all travelers surveyed; the two circular regions represent the two groups of travelers to Africa and Asia; and the shaded region represents the subset of those who have traveled to both continents.

(a) The set described here is represented by *that part of the left circle that is not shaded*. This description suggests that the answer can be found by taking the shaded part away from the first circle—in effect, subtracting the 70 from the 93, to get 23 travelers who have traveled to Africa but not to Asia.

(b) The set described here is represented by that part of the rectangle that is *in at least one of the two circles*. This description suggests adding the two numbers 93 and 155. But the 70 travelers who have traveled to both continents would be counted twice in the sum $93 + 155$. To correct the double counting, subtract 70 from the sum so that these 70 travelers are counted only once:

$$93 + 155 - 70 = 178.$$

(c) The set described here is represented by that part of the rectangle that is *not in either circle*. Let N be the number of these travelers. Note that the entire rectangular region has two main nonoverlapping parts: the part *outside* the circles and the part *inside* the circles. The first part represents N travelers and the second part represents $93 + 155 - 70 = 178$ travelers (from question (b)). Therefore,

$$250 = N + 178,$$

and solving for N yields

$$N = 250 - 178 = 72.$$

DATA ANALYSIS EXERCISES

(Answers on page 69)

1. The daily temperatures, in degrees Fahrenheit, for 10 days in May were 61, 62, 65, 65, 65, 68, 74, 74, 75, and 77.
   (a) Find the mean, median, and mode for the temperatures.
   (b) If each day had been 7 degrees warmer, what would have been the mean, median, and mode for those 10 measurements?

2. The ages, in years, of the employees in a small company are 22, 33, 21, 28, 22, 31, 44, and 19.
   (a) Find the mean, median, and mode for the 8 ages.
   (b) Find the range and standard deviation for the 8 ages.
   (c) If each of the employees had been 10 years older, what would have been the range and standard deviation of their ages?

3. A group of 20 values has mean 85 and median 80. A different group of 30 values has mean 75 and median 72.
   (a) What is the mean of the 50 values?
   (b) What is the median of the 50 values?

4. Find the mean, median, mode, range, and standard deviation for x, given the frequency distribution below.

x	f
0	2
1	6
2	3
3	2
4	4

5. In the frequency distribution below, y represents age on last birthday for 40 people. Find the mean, median, mode, and range for y.

y	f
17	2
18	7
19	19
20	9
21	2
22	0
23	1

6. How many different ways can the letters in the word STUDY be ordered?

7. Martha invited 4 friends to go with her to the movies. There are 120 different ways in which they can sit together in a row. In how many of those ways is Martha sitting in the middle?

8. How many 3-digit positive integers are odd and do not contain the digit “5”?

9. From a box of 10 light bulbs, 4 are to be removed. How many different sets of 4 bulbs could be removed?

10. A talent contest has 8 contestants. Judges must award prizes for first, second, and third places. If there are no ties, (a) in how many different ways can the 3 prizes be awarded, and (b) how many different groups of 3 people can get prizes?

11. If the probability is 0.78 that Marshall will be late for work at least once next week, what is the probability that he will not be late for work next week?

12. If an integer is randomly selected from all positive 2-digit integers (i.e., the integers 10, 11, 12, . . . , 99), find the probability that the integer chosen has

(a) a "4" in the tens place

(b) at least one "4"

(c) no "4" in either place

13. In a box of 10 electrical parts, 2 are defective.

(a) If one part is chosen randomly from the box, what is the probability that it is not defective?

(b) If two parts are randomly chosen from the box, without replacement, what is the probability that both are defective?

14. The table shows the distribution of a group of 40 college students by gender and class.

	Sophomores	Juniors	Seniors
Males	6	10	2
Females	10	9	3

If one student is randomly selected from this group, find the probability that the student chosen is

(a) not a junior

(b) a female or a sophomore

(c) a male sophomore or a female senior

15. $P(A \text{ or } B) = 0.60$ and $P(A) = 0.20$.

(a) Find $P(B)$ given that events A and B are mutually exclusive.

(b) Find $P(B)$ given that events A and B are independent.

16. Lin and Mark each attempt independently to decode a message. If the probability that Lin will decode the message is 0.80, and the probability that Mark will decode the message is 0.70, find the probability that

(a) both will decode the message

(b) at least one of them will decode the message

(c) neither of them will decode the message

17.

AVERAGE AND HIGH WIND SPEED FOR SELECTED STATIONS OVER A 10-YEAR PERIOD (1971-80) (miles per hour)		
Station	Average	High
Atlanta, GA	9.1	71
Boston, MA	12.6	65
Buffalo, NY	12.3	91
Chicago, IL	10.4	60
Cincinnati, OH	7.1	49
Denver, CO	9.0	56
Miami, FL	9.2	132
Montgomery, AL	6.7	72
New York, NY	9.4	70
Omaha, NE	10.8	109
Pittsburgh, PA	9.3	58
San Diego, CA	6.7	51
Washington, DC	9.3	78

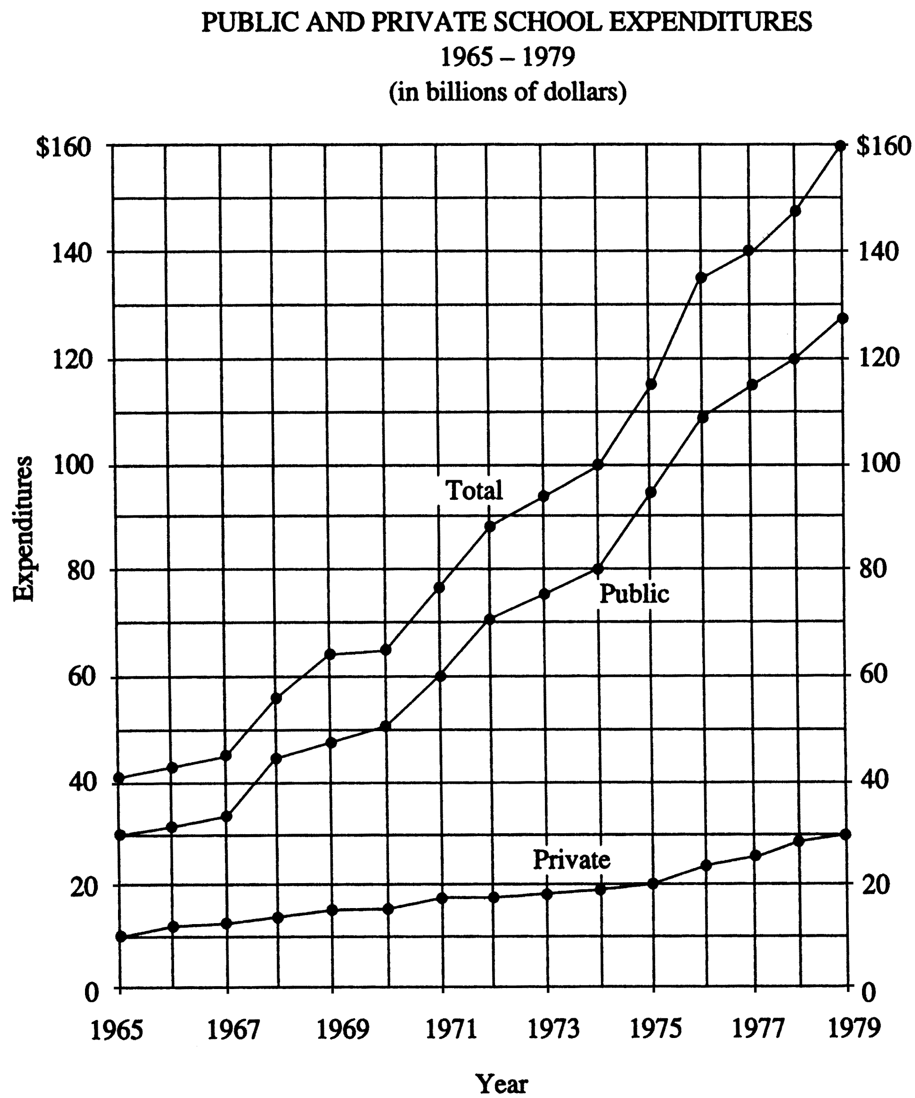
SPEED AND OFFICIAL DESIGNATIONS OF WINDS	
Designation	Miles per Hour
Calm	Less than 1
Light air	1 to 3
Light breeze	4 to 7
Gentle breeze	8 to 12
Moderate breeze	13 to 18
Fresh breeze	19 to 24
Strong breeze	25 to 31
Near gale	32 to 38
Gale	39 to 46
Strong gale	47 to 54
Storm	55 to 63
Violent storm	64 to 73
Hurricane	74 and above

(a) Which station has a high wind speed that is the median of the high wind speeds for all the stations listed?

(b) For those stations that have recorded hurricane winds at least once during the 10-year period, what is the arithmetic mean of their average wind speeds?

(c) For how many of the stations is the ratio of high wind speed to average wind speed greater than 10 to 1 ?

18.

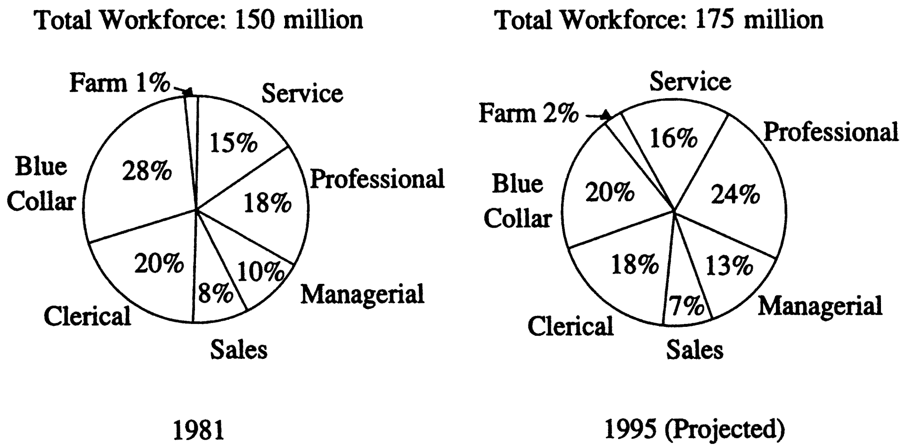


(a) In which year did total expenditures increase the most from the year before?

(b) In 1979 private school expenditures were approximately what percent of total expenditures?

19.

DISTRIBUTION OF WORKFORCE BY OCCUPATIONAL CATEGORY FOR COUNTRY *X* IN 1981 AND PROJECTED FOR 1995



(a) In 1981, how many categories each comprised more than 25 million workers?

(b) What is the ratio of the number of workers in the Professional category in 1981 to the projected number of such workers in 1995 ?

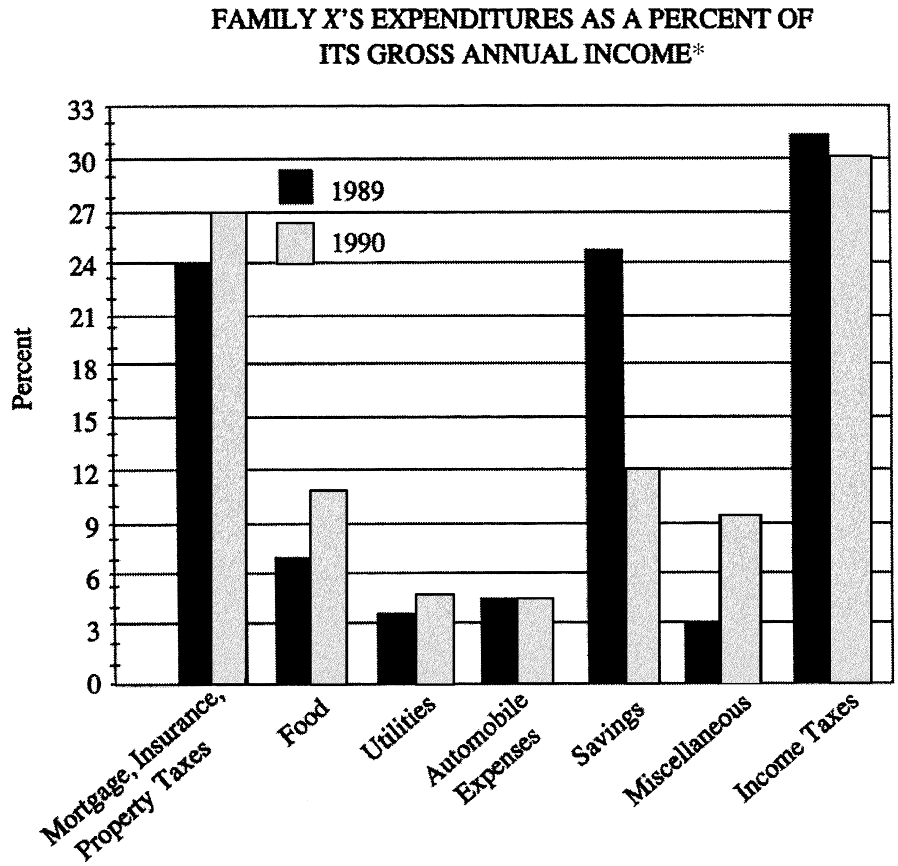
(c) From 1981 to 1995, there is a projected increase in the number of workers in which of the following categories?

I. Sales

II. Service

III. Clerical

20.



*1989 Gross annual income: \$50,000
1990 Gross annual income: \$45,000

(a) In 1989 Family X used a total of 49 percent of its gross annual income for two of the categories listed. What was the total amount of Family X's income used for those same categories in 1990 ?

(b) Family X's gross income is the sum of Mr. X's income and Mrs. X's income. In 1989 Mr. and Mrs. X each had an income of \$25,000. If Mr. X's income increased by 10 percent from 1989 to 1990, by what percent did Mrs. X's income decrease for the same period?

ANSWERS TO DATA ANALYSIS EXERCISES

1. (a) mean = 68.6, median = 66.5, mode = 65
   (b) Each measure would have been 7 degrees greater.

2. (a) mean = 27.5, median = 25, mode = 22
   (b) range = 25, standard deviation $\approx$ 7.8
   (c) range = 25, standard deviation $\approx$ 7.8

3. (a) mean = 79
   (b) The median cannot be determined from the information given.

4. mean = 2, median = 2, mode = 1, range = 4, standard deviation $\approx$ 1.4

5. mean = 19.15, median = 19, mode = 19, range = 6

6. 120

7. 24

8. 288

9. 210

10. (a) 336 (b) 56

11. 0.22

12. (a) $\frac{1}{9}$ (b) $\frac{1}{5}$ (c) $\frac{4}{5}$

13. (a) $\frac{4}{5}$ (b) $\frac{1}{45}$

14. (a) $\frac{21}{40}$ (b) $\frac{7}{10}$ (c) $\frac{9}{40}$

15. (a) 0.40 (b) 0.50

16. (a) 0.56 (b) 0.94 (c) 0.06

17. (a) New York (b) 10.4 (c) Three

18. (a) 1976 (b) 19%

19. (a) Three (b) 9 to 14, or 9/14 (c) I, II, and III

20. (a) $17,550 (b) 30%