The Algebra of the Heckscher-Ohlin Neoclassical Trade Model

Let there be two goods – corn and manufactures (c and m), two factors – capital and labor (K and L) and two countries – Home and Foreign. Assume that all input (factor) and output markets are competitive. Let production technologies be identical across countries. Let the production of good $i \in \{c, m\}$ be denoted $Q_i = F_i(K_i, L_i)$. There is no joint production. This means that a factor used in the production of one good cannot be used simultaneously in the production of the other good. Thus, factors must be allocated distinctly between the production of the two goods. Let $F_i(\cdot)$ be:

1. linearly homogenous (constant-returns-to-scale). Thus, $zQ_i = F_i(zK_i, zL_i)$ for $z \in \Re$.
2. concave for all i. Thus, we know that: $F_K, F_L > 0$; $F_{KK}, F_{LL} < 0$; and $F_{KL} > 0$ (by linear homogeneity).

Further, $F_i(\cdot)$ fulfills the Inada conditions, so that $F_i(0, L) = F_i(0, L) = 0$, $F_K(0, L) = F_L(K, 0) = \infty$, and $F_K(\infty, L) = F_L(K, \infty) = 0$. The following identities hold: $K_c + K_m = K$, and $L_c + L_m = L$.

With linear homogeneity, we can rewrite the production function in intensive form: $q_i = \frac{Q_i}{L_i} = F_i\left(\frac{K_i}{L_i}, 1\right) = f_i(\frac{K_i}{L_i}) = f_i(k_i)$, where $k_i = \frac{K_i}{L_i}$ and $q_i = \frac{Q_i}{L_i}$. Let P_i denote the price of good i. Let w denote the wage rate (return to L) and r denote the rental rate (return to K).

With the above assumptions, we can see that:

$$\begin{aligned}
\frac{\frac{\partial Q_i}{\partial L_i}}{L_i} - \frac{Q_i}{(L_i)^2} &= -f_i'(\frac{K_i}{L_i})\frac{K_i}{(L_i)^2} \\
\frac{\frac{\partial Q_i}{\partial L_i}}{L_i} - q_i\frac{1}{L_i} &= -f_i'(k_i)k_i\frac{1}{L_i} \\
\frac{\partial Q_i}{\partial L_i} &= q_i - f_i'(k_i)k_i \\
&= f_i(k_i) - f_i'(k_i)k_i.
\end{aligned}$$

We can also see that:

$$\frac{\frac{\partial Q_i}{\partial K_i}}{L_i} = f_i'(k_i)\frac{1}{L_i} \Rightarrow \frac{\partial Q_i}{\partial K_i} = f_i'(k_i) > 0.$$

Furthermore, the above implies that $f_i''(k_i) < 0$. The production function in intensive form is also concave.

1 Prices in the Heckscher-Ohlin Model

The wage and rental rates are:

$$\begin{aligned}
w &= P_i\frac{\partial Q_i}{\partial L_i} = P_i\left[f_i(k_i) - f_i'(k_i)k_i\right] \text{ and} \\
r &= P_i\frac{\partial Q_i}{\partial K_i} = P_i\left[f_i'(k_i)\right]
\end{aligned}$$

Implicitly, we are assuming *incomplete specialization*, since i denotes either c or m. This implies that the production cost ranges of the two countries must overlap. Otherwise, at least one of the countries will completely specialize when opened up for trade, and thus the above input market equilibrium conditions will not hold for both goods.

Let $\omega = \frac{w}{r}$ denote the relative factor price (the price of labor in terms of capital). Using the input market equilibrium conditions, we can see that:

$$\omega = \frac{P_i \left[f_i\left(k_i\right) - f_i'(k_i)k_i \right]}{P_i \left[f_i'\left(k_i\right) \right]} = \frac{f_i(k_i)}{f_i'(k_i)} - k_i$$

Using implicit differentiation of the above identity, we can see that:

$$\begin{aligned}
1 &= \frac{f_i'\left(k_i\right)}{f_i'\left(k_i\right)} \frac{\partial k_i}{\partial \omega} - \frac{f_i(k_i)}{\left[f_i'(k_i)\right]^2} f_i''\left(k_i\right) \frac{\partial k_i}{\partial \omega} - \frac{\partial k_i}{\partial \omega} \\
1 &= \frac{\partial k_i}{\partial \omega} - \frac{f_i(k_i) f_i''\left(k_i\right)}{\left[f_i'(k_i)\right]^2} \frac{\partial k_i}{\partial \omega} - \frac{\partial k_i}{\partial \omega} \\
\frac{\partial k_i}{\partial \omega} &= -\frac{\left[f_i'(k_i)\right]^2}{f_i(k_i) f_i''\left(k_i\right)} > 0, \text{ since } f_i'' < 0.
\end{aligned}$$

It is possible for $k_c(\omega)$ and $k_m(\omega)$ to cross in $k - \omega$ space. In this case, we cannot definitively distinguish the capital-intensive versus the labor-intensive good (determined by a comparison of the capital:labor ratios for the two goods). For some relative factor prices, the relationship between the capital:labor ratios of the two goods may switch. Such a crossover is referred to as a factor intensity reversal. Assume that there are *no factor intensity reversals*.

Let $P = \frac{P_m}{P_c}$ denote the relative output price (the price of manufactures in terms of corn). We can see then that:

$$\begin{aligned}
r &= P_c f_c'\left(k_c\right) = P_m f_m'\left(k_m\right) \Rightarrow \\
P &= \frac{f_c'\left(k_c\right)}{f_m'\left(k_m\right)}.
\end{aligned}$$

1.1 Stolper-Samuelson Theorem and Factor Price Equalization

Implicit differentiation of the relative output price equation reveals that:

$$\begin{aligned}
\frac{\partial P}{\partial \omega} &= \frac{f_c''(k_c)}{f_m'(k_m)}\frac{\partial k_c}{\partial \omega} - \frac{f_c'(k_c)}{[f_m'(k_m)]^2} f_m''(k_m)\frac{\partial k_m}{\partial \omega} \\
&= \frac{f_c''(k_c)}{f_m'(k_m)}\left\{-\frac{[f_c'(k_c)]^2}{f_c(k_c)f_c''(k_c)}\right\} + \frac{f_c'(k_c)}{[f_m'(k_m)]^2} f_m''(k_m)\left\{\frac{[f_m'(k_m)]^2}{f_m(k_m)f_m''(k_m)}\right\} \\
&= \frac{f_c'(k_c)}{f_m(k_m)} - \frac{[f_c'(k_c)]^2}{f_m'(k_m)\, f_c(k_c)} \\
&= \frac{f_c'(k_c)}{f_m(k_m)} - P\left[\frac{f_c'(k_c)}{f_c(k_c)}\right] \Rightarrow \\
\frac{\partial P}{\partial \omega}\frac{1}{P} &= \frac{f_m'(k_m)\, f_c'(k_c)}{f_c'(k_c)\, f_m(k_m)} - \frac{f_c'(k_c)}{f_c(k_c)} \\
&= \frac{f_m'(k_m)}{f_m(k_m)} - \frac{f_c'(k_c)}{f_c(k_c)} \\
&= \frac{1}{\omega + k_m} - \frac{1}{\omega + k_c}, \text{ by the definition of } \omega \Rightarrow \\
\frac{\partial P}{\partial \omega}\frac{\omega}{P} &= \frac{\omega\left[(\omega + k_c) - (\omega + k_m)\right]}{(\omega + k_m)(\omega + k_c)} = \frac{\omega(k_c - k_m)}{(\omega + k_m)(\omega + k_c)} \gtreqless 0, \text{ as } k_c \gtreqless k_m.
\end{aligned}$$

Thus, $sgn(\frac{\partial P}{\partial \omega})$ depends on the relative factor intensities. If c is labor-intensive ($k_c < k_m$), then an increase in the price of manufactures in terms of corn means that the factor price of capital in terms of labor increases. As P rises, ω falls; or, $\frac{\partial P}{\partial \omega} < 0$. Thus, a rise in the relative price of the capital-intensive good generates a rise in the relative factor price of capital (which is $\frac{1}{\omega}$). Similarly, a rise in the relative price of the labor-intensive good generates a rise in the relative factor price of labor. This is the *Stolper-Samuelson* result.

With factor intensity non-reversal, there is a one-to-one relationship between P and ω (implied by the inverse function theorem and the monotonicity of the partial derivative). Under free trade then, the existence of common commodity prices, $P_c = P_c^*$ and $P_m = P_m^*$, implies that there will be relative factor price equalization, with $\omega = \omega^*$. The asterisk denotes Foreign variables. In fact, absolute *factor price equalization* will occur, since:

$$\begin{aligned}
r^* &= P_c^* f_c'^*(k_c^*(\omega^*)) \\
&= P_c f_c'^*(k_c^*(\omega)) \\
&= P_c f_c'(k_c(\omega)) = r.
\end{aligned}$$

Demand functions are assumed to be homothetic and identical across countries. This means that the ratios of goods demanded depends only upon relative prices and nothing else. Under these conditions and common commodity prices, specialization in a good implies export of that good.

Absolute factor price equalization allows for a stronger Stolper-Samuelson result, which can be stated in terms of price changes and not merely relative price changes. A factor's price rises if the output price of the good which is intensive in that factor rises. The relative sizes of the price effects can be found. Under the assumption that unit-labor and unit-capital requirements are constant, the zero-profit condition implied by perfect competition means that:

$$\begin{aligned} P_m &= wa_{m,L} + ra_{m,K} \text{ and } P_c = wa_{c,L} + ra_{c,K} \Rightarrow \\ \widehat{P_m} &= \widehat{w}a_{m,L} + \widehat{r}a_{m,K} \text{ and } \widehat{P_c} = \widehat{w}a_{c,L} + \widehat{r}a_{c,K}, \end{aligned}$$

where hats denote growth rates and $a_{i,j}$ is the unit-factor requirement for good i in factor j. Suppose that $\widehat{P_c} = 0 < \widehat{P_m}$; the price of manufactures rises, while the price of corn stays constant. Since manufactures is capital-intensive, we know that $\frac{a_{m,K}}{a_{m,L}} > \frac{a_{c,K}}{a_{c,L}}$. The rental rate r thus receives a larger weight in the above equation for the price of manufactures than does the wage rate w, compared to the equation for the price of corn. In order then for the price of corn to remain constant while the price of manufactures rises, the wage rate must fall while the rental rate rises proportionately more than does the price of manufactures. This is expressed here:

$$\widehat{w} < \widehat{P_c} = 0 < \widehat{P_m} < \widehat{r}.$$

This is sometimes referred to as the magnification effect of prices in the Heckscher-Ohlin model. The range of factor price changes is larger than the range of output price changes.

2 Heckscher-Ohlin Theorem and the Pattern of Trade

Given that production technologies are identical across countries, then all that distinguishes Home and Foreign are their factor endowments. Autarky factor endowments will drive autarky relative factor prices. These relative factor prices in turn determine the relative output price, P. Once trade is opened, the countries will specialize (incompletely) in the production of the good which has the lower opportunity cost. For example, suppose that $P^A > P^{A*}$, where the A denotes autarky quantities. Then, free trade implies that $P^A > P = P^* > P^{A*}$. From the autarky relative output prices, we know that Home will specialize in the production of c and Foreign will specialize in the production of m (since Home has a higher autarky relative output price of manufactures in terms of corn). Suppose that c is labor-intensive and m is capital-intensive ($k_m > k_c$). The reason Foreign's autarky relative output price is lower than Home's price is that $\omega^{A*} > \omega^A$, which is driven by Foreign being relatively abundant in capital and Home being relatively abundant in labor. Hence, autarky factor endowments determine the pattern of trade. This is the fundamental result of the *Heckscher-Ohlin* model.

3 Rybczynski Theorem and the Biased Expansion of the Production Possibilities Frontier

Rybczynski discovered that the Heckscher-Ohlin model implies that an expansion of a factor (e.g., $L^1 > L^0$) generates a more than proportionate rise in output of the good intensive in that factor, given that commodity prices are constant. Similarly, the good intensive in the non-expanding factor suffers a decline in output. As above, assume that c is labor-intensive and m is capital-intensive. Recall that:

$$\begin{aligned} K &= K_c + K_m \\ \frac{K}{L} &= \frac{K_c}{L} + \frac{K_m}{L} \\ k &= k_c \frac{L_c}{L} + k_m \frac{L_m}{L} \\ &= k_c l_c + k_m l_m, \text{ where } l_c = \frac{L_c}{L} \text{ and } l_m = \frac{L_m}{L}. \end{aligned}$$

Then, if K rises, k must rise. At constant commodity prices, factor intensity non-reversal means that the relative factor price ω is constant. Hence, k_m and k_c are constant. Thus, to ensure that the above identity holds, the weights (l_c and l_m) must readjust. The capital-intensive good must receive a higher weight than previously. Hence, l_c must fall. So, L_c must fall and thus K_c must fall. This finally implies that Q_c must fall. This is the classic *Rybczynski* effect.

The range of implied variable changes can also be demonstrated. Let $K = a_{c,K}Q_c + a_{m,K}Q_m$, where $a_{i,K} = \frac{K_i}{Q_i}$ represents the constant unit-capital requirement for good i. From this identity, we know that: $\widehat{K} = \frac{K_c}{K}\widehat{Q_c} + \frac{K_m}{K}\widehat{Q_m}$, where hats denote growth rates (e.g., $\widehat{K} = \frac{dK}{K}$). Suppose that $\widehat{L} = 0 < \widehat{K}$; the capital stock expands while the labor supply remains constant. As seen above, output of the labor-intensive good must fall, which means that $\widehat{Q_c} < 0$. The relationship of the growth rates then implies that:

$$\widehat{Q_c} < 0 = \widehat{L} < \widehat{K} < \widehat{Q_m},$$

since the unit-capital requirements are held constant. Similar to the Stolper-Samuelson theorem's magnification effect of prices, the range of output changes is wider than the range of factor endowment changes.

4 Duality of the Stolper-Samuelson and Rybczynski Theorems

There is a duality relationship between the Rybczynski theorem and the Stolper-Samuelson theorem. Given perfectly competitive markets, we know that national income is given by:

$$Y(P_c, P_m, K, L) = P_c Q_c + P_m Q_m = wL + rK.$$

From this relationship, we see that:

$$\begin{aligned} \frac{\partial Y}{\partial P_m} &= \left[P_c \frac{\partial Q_c}{\partial P_m} + P_m \frac{\partial Q_m}{\partial P_m} \right] + Q_m \\ &= Q_m, \text{ by the envelope theorem (the term in brackets must be zero).} \end{aligned}$$

We also know that: $\frac{\partial Y}{\partial K} = r$. Combining these two results, we can see that the cross derivative must be:

$$\frac{\partial^2 Y}{\partial P_m \partial K} = \frac{\partial Q_m}{\partial K} = \frac{\partial r}{\partial P_m}.$$

Thus, the Rybczynski effect $\frac{\partial Q_m}{\partial K}$ is shown to be the same as the Stolper-Samuelson effect $\frac{\partial r}{\partial P_m}$.

5 Summary of Results of the Heckscher-Ohlin Model

Assuming the 2x2x2 structure of the economy as above, with c labor-intensive and m capital-intensive, the results of the Heckscher-Ohlin model are summarized in the following table:

Heckscher-Ohlin Model Predictions	
Theorem	Implication
Heckscher-Ohlin	Home is relatively labor abundant $\Rightarrow$ Home exports c.
Stolper-Samuelson	$P_c \uparrow \Rightarrow w \uparrow$ and $\widehat{r} < \widehat{P_m} = 0 < \widehat{P}_c < \widehat{w}$.
Factor Price Equalization	$P_c = P_c^*,\ P_m = P_m^* \Rightarrow w = w^*,\ r = r^*$.
Rybczynski	$L \uparrow \Rightarrow Q_c \uparrow$ and $\widehat{Q_m} < 0 = \widehat{K} < \widehat{L} < \widehat{Q_c}$.
Duality	$\frac{\partial Q_c}{\partial L} = \frac{\partial w}{\partial P_c}$.