The Human Capital Augmented Solow Model

In a 1992 article in the *Quarterly Journal of Economics*, Mankiw, Romer, and Weil presented the human capital augmented Solow model of economic growth. They noted that such a model fit the data extremely well. Here, we'll derive and go over the steady-state implications of the model, and then compare these results to the standard Solow model conclusions.

Assume that the economy produces one good, output (Y). It is produced according to:

$$Y(t) = K(t)^{\alpha} H(t)^{\beta} [A(t) L(t)]^{1-\alpha-\beta},$$

where $\alpha, \beta \in [0,1]$, $\alpha + \beta \in [0,1]$, and t denotes time. This implies that the production function exhibits constant returns to scale in its three factors: physical capital (K), human capital (H), and productivity-augmented labor (AL). Specifically, it is a Cobb-Douglas production function. All markets (both input and output markets) are assumed to be perfectly competitive. All firms are assumed to be identical. The economy can then be described by a representative agent.

Physical capital and human capital are assumed to be accumulating factors; i.e., the representative agent saves output to have more capital (either physical or human). Their equations of motion are:

$$\begin{aligned}
\dot{K}(t) &= s_K Y(t) - \delta K(t) \\
\dot{H}(t) &= s_H Y(t) - \delta H(t),
\end{aligned}$$

where s_K and s_H are the saving rates for physical capital and human capital respectively. They are exogenously given. Notice that both physical capital and human capital are assumed to depreciate at the same rate, δ. This will make our lives much easier later, as it simplifies the algebra tremendously.

The equations of motion for labor (L) and labor-augmenting productivity (A) are:

$$\dot{L}(t) = nL(t) \ \text{ and } \ \dot{A}(t) = gA(t),$$

where n and g are exogenously given growth rates.

With these five equations, we can solve for the balanced growth paths of output, physical capital, and human capital.[1] The trick here is to find some transformation of these variables which converges to a steady-state. In the Solow model, we transform the system so that everything is expressed in per "effective" worker terms. This means that we divide each variable by $A(t) L(t)$, or the number of effective workers (productivity-augmented workers) in the economy at time t. This is also called putting the system into *intensive* form. We'll follow the same strategy here. Define $\tilde{y}(t) = \frac{Y(t)}{A(t)L(t)}$, $\tilde{k}(t) = \frac{K(t)}{A(t)L(t)}$, and $\tilde{h}(t) = \frac{H(t)}{A(t)L(t)}$.

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[1]To get an exact solution for the levels of these variables at any point in time, we also need initial conditions for physical capital, human capital, productivity, and labor. However, to find the entire time path for these variables requires a knowledge of linear differential equation solutions, which we have not covered in this course. Here, the focus will be on the long run growth of these variables and their long run levels.

In intensive form, the production function and equations of motion for physical and human capital become:

$$\frac{Y(t)}{A(t)L(t)} = \frac{K(t)^{\alpha}H(t)^{\beta}[A(t)L(t)]^{1-\alpha-\beta}}{A(t)L(t)}$$

$$\tilde{y}(t) = \frac{K(t)^{\alpha}H(t)^{\beta}[A(t)L(t)]^{1-\alpha-\beta}}{[A(t)L(t)]^{\alpha}[A(t)L(t)]^{\beta}[A(t)L(t)]^{1-\alpha-\beta}}$$

$$\tilde{y}(t) = \tilde{k}(t)^{\alpha}\tilde{h}(t)^{\beta}$$

$$\dot{\tilde{k}}(t) = \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{[A(t)L(t)]^{2}}\left[\dot{A}(t)L(t) + A(t)\dot{L}(t)\right]$$

$$= \frac{s_{K}Y(t) - \delta K(t)}{A(t)L(t)} - \frac{K(t)}{A(t)L(t)}\frac{\left[\dot{A}(t)L(t) + A(t)\dot{L}(t)\right]}{A(t)L(t)}$$

$$= s_{K}\tilde{y}(t) - \delta\tilde{k}(t) - \tilde{k}(t)[g+n]$$

$$\dot{\tilde{k}}(t) = s_{K}\tilde{y}(t) - [n+g+\delta]\tilde{k}(t)$$

$$\dot{\tilde{h}}(t) = \frac{\dot{H}(t)}{A(t)L(t)} - \frac{H(t)}{[A(t)L(t)]^{2}}\left[\dot{A}(t)L(t) + A(t)\dot{L}(t)\right]$$

$$= \frac{s_{H}Y(t) - \delta H(t)}{A(t)L(t)} - \frac{H(t)}{A(t)L(t)}\frac{\left[\dot{A}(t)L(t) + A(t)\dot{L}(t)\right]}{A(t)L(t)}$$

$$= s_{H}\tilde{y}(t) - \delta\tilde{h}(t) - \tilde{h}(t)[g+n]$$

$$\dot{\tilde{h}}(t) = s_{H}\tilde{y}(t) - [n+g+\delta]\tilde{h}(t).$$

In a steady-state, physical and human capital per effective worker must be constant. This implies that we can solve for the steady-state by finding the values for $\tilde{k}$ and $\tilde{h}$ which set the above equations of motion to zero (other than the trivial steady-state given by setting either $\tilde{k}$ or $\tilde{h}$ equal to zero). The steady-state conditions are then:

$$s_{K}\tilde{y}(t) = [n+g+\delta]\tilde{k}(t)$$

$$s_{H}\tilde{y}(t) = [n+g+\delta]\tilde{h}(t).$$

Of course, we also need the production function definition which holds at all points in time, $\tilde{y}(t) = \tilde{k}(t)^{\alpha}\tilde{h}(t)^{\beta}$. We can substitute this production function into the above two equations. With two equations and two unknowns ($\tilde{k}$ and $\tilde{h}$), we can find the exact solution for this system. First, we solve for one of the variables in terms of the other. Let's solve for $\tilde{h}$ in terms of $\tilde{k}$.

$$s_{H}\tilde{k}(t)^{\alpha}\tilde{h}(t)^{\beta} = [n+g+\delta]\tilde{h}(t)$$

$$\tilde{h}(t)^{\beta-1} = \left[\frac{n+g+\delta}{s_{H}}\right]\tilde{k}(t)^{-\alpha}$$

$$\tilde{h}(t) = \left[\frac{s_{H}}{n+g+\delta}\right]^{\frac{1}{1-\beta}}\tilde{k}(t)^{\frac{\alpha}{1-\beta}}.$$

Then, we substitute this expression into the other steady-state condition, and solve for $\tilde{k}$.

$$s_K \tilde{k}(t)^\alpha \left[\left[\frac{s_H}{n+g+\delta} \right]^{\frac{1}{1-\beta}} \tilde{k}(t)^{\frac{\alpha}{1-\beta}} \right]^\beta = [n+g+\delta] \tilde{k}(t)$$

$$\tilde{k}(t)^{\alpha-1} \left[\frac{s_H}{n+g+\delta} \right]^{\frac{\beta}{1-\beta}} \tilde{k}(t)^{\frac{\alpha\beta}{1-\beta}} = \left[\frac{n+g+\delta}{s_K} \right]$$

$$\tilde{k}(t)^{\frac{(\alpha-1)(1-\beta)}{1-\beta} + \frac{\alpha\beta}{1-\beta}} = \left[\frac{s_K}{n+g+\delta} \right]^{-1} \left[\frac{s_H}{n+g+\delta} \right]^{\frac{-\beta}{1-\beta}}$$

$$\tilde{k}(t)^{\frac{\alpha-\alpha\beta-1+\beta+\alpha\beta}{1-\beta}} = \left[\frac{s_K}{n+g+\delta} \right]^{-1} \left[\frac{s_H}{n+g+\delta} \right]^{\frac{-\beta}{1-\beta}}$$

$$\tilde{k}(t)^{\frac{\alpha+\beta-1}{1-\beta}} = \left[\frac{s_K}{n+g+\delta} \right]^{-1} \left[\frac{s_H}{n+g+\delta} \right]^{\frac{-\beta}{1-\beta}}$$

$$\tilde{k}^*(t) = \left[\frac{s_K}{n+g+\delta} \right]^{-\left(\frac{1-\beta}{\alpha+\beta-1}\right)} \left[\frac{s_H}{n+g+\delta} \right]^{\left(\frac{-\beta}{1-\beta}\right)\left(\frac{1-\beta}{\alpha+\beta-1}\right)}$$

$$\tilde{k}^*(t) = \left(\frac{s_K}{n+g+\delta} \right)^{\frac{1-\beta}{1-\alpha-\beta}} \left(\frac{s_H}{n+g+\delta} \right)^{\frac{\beta}{1-\alpha-\beta}}.$$

The asterisk denotes the steady-state value of a variable. Now, we can substitute this back into our expression for $\tilde{h}$.

$$\tilde{h}^*(t) = \left[\frac{s_H}{n+g+\delta} \right]^{\frac{1}{1-\beta}} \left[\left[\frac{s_K}{n+g+\delta} \right]^{\left(\frac{1-\beta}{1-\alpha-\beta}\right)} \left[\frac{s_H}{n+g+\delta} \right]^{\left(\frac{\beta}{1-\alpha-\beta}\right)} \right]^{\frac{\alpha}{1-\beta}}$$

$$= \left[\frac{s_H}{n+g+\delta} \right]^{\frac{1}{1-\beta}} \left[\frac{s_K}{n+g+\delta} \right]^{\frac{\alpha(1-\beta)}{(1-\alpha-\beta)(1-\beta)}} \left[\frac{s_H}{n+g+\delta} \right]^{\frac{\alpha\beta}{(1-\alpha-\beta)(1-\beta)}}$$

$$= \left[\frac{s_H}{n+g+\delta} \right]^{\frac{1-\alpha-\beta}{(1-\alpha-\beta)(1-\beta)} + \frac{\alpha\beta}{(1-\alpha-\beta)(1-\beta)}} \left[\frac{s_K}{n+g+\delta} \right]^{\frac{\alpha}{(1-\alpha-\beta)}}$$

$$= \left[\frac{s_H}{n+g+\delta} \right]^{\frac{1-\alpha-\beta+\alpha\beta}{(1-\alpha-\beta)(1-\beta)}} \left[\frac{s_K}{n+g+\delta} \right]^{\frac{\alpha}{(1-\alpha-\beta)}}$$

$$= \left[\frac{s_H}{n+g+\delta} \right]^{\frac{1-\alpha-\beta(1-\alpha)}{(1-\alpha-\beta)(1-\beta)}} \left[\frac{s_K}{n+g+\delta} \right]^{\frac{\alpha}{(1-\alpha-\beta)}}$$

$$= \left[\frac{s_H}{n+g+\delta} \right]^{\frac{(1-\beta)(1-\alpha)}{(1-\alpha-\beta)(1-\beta)}} \left[\frac{s_K}{n+g+\delta} \right]^{\frac{\alpha}{(1-\alpha-\beta)}}$$

$$\tilde{h}^*(t) = \left(\frac{s_H}{n+g+\delta} \right)^{\frac{1-\alpha}{1-\alpha-\beta}} \left(\frac{s_K}{n+g+\delta} \right)^{\frac{\alpha}{1-\alpha-\beta}}.$$

With these expressions for $\tilde{k}^*$ and $\tilde{h}^*$, we can now solve for $\tilde{y}^*$.

$$
\begin{aligned}
\tilde{y}^*(t) &= \tilde{k}^*(t)^\alpha \, \tilde{h}^*(t)^\beta \\[2mm]
&= \left[\left(\frac{s_K}{n+g+\delta} \right)^{\frac{1-\beta}{1-\alpha-\beta}} \left(\frac{s_H}{n+g+\delta} \right)^{\frac{\beta}{1-\alpha-\beta}} \right]^\alpha \left[\left(\frac{s_H}{n+g+\delta} \right)^{\frac{1-\alpha}{1-\alpha-\beta}} \left(\frac{s_K}{n+g+\delta} \right)^{\frac{\alpha}{1-\alpha-\beta}} \right]^\beta \\[2mm]
&= \left(\frac{s_K}{n+g+\delta} \right)^{\frac{(1-\beta)\alpha}{1-\alpha-\beta}} \left(\frac{s_H}{n+g+\delta} \right)^{\frac{\alpha\beta}{1-\alpha-\beta}} \left(\frac{s_H}{n+g+\delta} \right)^{\frac{(1-\alpha)\beta}{1-\alpha-\beta}} \left(\frac{s_K}{n+g+\delta} \right)^{\frac{\alpha\beta}{1-\alpha-\beta}} \\[2mm]
&= \left(\frac{s_K}{n+g+\delta} \right)^{\frac{\alpha-\alpha\beta+\alpha\beta}{1-\alpha-\beta}} \left(\frac{s_H}{n+g+\delta} \right)^{\frac{\beta-\alpha\beta+\alpha\beta}{1-\alpha-\beta}} \\[2mm]
\tilde{y}^*(t) &= \left(\frac{s_K}{n+g+\delta} \right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{s_H}{n+g+\delta} \right)^{\frac{\beta}{1-\alpha-\beta}} .
\end{aligned}
$$

The standard Solow model results can be recovered from the above system by imposing the restriction that $\beta = 0$. In the standard Solow model, the steady-state level of output per effective worker is:

$$
\tilde{y}^*_{\text{Solow}}(t) = \left(\frac{s_K}{n+g+\delta} \right)^{\frac{\alpha}{1-\alpha}} .
$$

Notice the similarity of the two results. When $\beta \neq 0$, the rate of human capital accumulation can affect the steady-state level of output per effective worker. The general message though is the same – the more that is saved, the higher will be the level of output per effective worker. From an empirical perspective, the addition of human capital to the model allows for another dimension to be invoked in explaining differences in output levels across countries. Countries who invest in education are predicted to have higher income levels than those who don't, for any given investment rate in physical capital.

Since output per worker on the balanced growth path is $\left(\frac{Y(t)}{L(t)} \right)^* = A(t)\, \tilde{y}^*(t)$ regardless of whether or not human capital is included, the growth of output per worker on the balanced growth path remains g, the rate of technological progress or the growth rate of labor-augmenting productivity. Growth of output per worker on the balanced growth path in the human capital augmented Solow model is the same as in the standard model. It is:

$$
\frac{\left(\dfrac{\dot{Y(t)}}{L(t)} \right)}{\left(\dfrac{Y(t)}{L(t)} \right)} = g.
$$

Since all countries draw upon the same stock of technology, the model predicts similar long run growth experiences for all countries. However, the addition of human capital to the model increases our ability to explain cross-country differences in income levels.