The Lucas Imperfect Information Model

Based on the work of Lucas (1972) and Phelps (1970), the imperfect information model represents an important milestone in modern economics. The essential idea of the model is that producers' inability to distinguish between price movements due to relative price changes (to which they should respond) and aggregate price level changes (e.g., general inflation, to which they should not respond) generates an upward-sloping aggregate supply curve. Producers' attribute some proportion of any observed aggregate price level change to a relative price change, and thus change the quantity of goods that they produce. First, we will solve the model assuming perfect information about price changes, and then solve it assuming imperfect information about price changes.

1 The Lucas Island Model under Perfect Information

Islands consume a market basket of produced goods (denoted C), which has unit price P (this will be a price index of all produced goods). Production cannot be saved, but must be consumed instantaneously. Production is denoted Y. Individuals on islands work to receive income to purchase the market basket, with island labor input denoted L. Production rises one-for-one with the amount of labor input. Let z index islands. Let the island representative agent utility maximization problem be:

$$\begin{aligned}
\max_{C(z),L(z)} U(z) &= C(z) - \frac{1}{\gamma} L(z)^{\gamma} \\
\text{s.t. } C(z) &= \frac{P(z)\,Y(z)}{P} \\
Y(z) &= L(z) \\
\text{and } \gamma &> 1.
\end{aligned}$$

The marginal utility of consumption is constant. Combined with the budget constraint, this implies that islands want to maximize their income to maximize their consumption. However, they experience an increasing marginal disutility of work. They thus must balance their marginal utility of consumption with their marginal disutility of work. The individual optimization problem then gives us:

$$\begin{aligned}
U(z) &= \frac{P(z)\,L(z)}{P} - \frac{1}{\gamma} L(z)^{\gamma} \\
&\Rightarrow \frac{\partial U(z)}{\partial L(z)} = \frac{P(z)}{P} - L(z)^{\gamma-1} = 0 \\
&\Rightarrow Y(z) = L(z) = \left(\frac{P(z)}{P}\right)^{\frac{1}{\gamma-1}} \\
&\Rightarrow y(z) = \frac{1}{\gamma-1}\left(p(z) - p\right).
\end{aligned}$$

Lowercase letters denote the logarithmic values of the corresponding uppercase letter variables. Because goods are not storable, solution of the instantaneous problem gives us the solution for production at any point in time.

Let $\alpha = \frac{1}{\gamma-1} > 0$. Then, the island supply function at time t is given by:

$$y_t(z) = \alpha\left[p_t(z) - p_t\right].$$

Suppose that the island has a demand function for its production at time t given by:

$$y_t^d(z) = m_t(z) - p_t(z).$$

Here, we can think of m as denoting the log money supply for island z. It is actually just a shorthand for modeling the demand side. If we apply a market clearing condition, then we have:

$$\begin{aligned} y_t(z) &= y_t^d(z) \\ \alpha\left[p_t(z) - p_t\right] &= m_t(z) - p_t(z) \\ p_t(z) &= \frac{m_t(z) + \alpha p_t}{1+\alpha}. \end{aligned}$$

Suppose that there are N total islands. Aggregation across islands requires that $m_t = p_t$, where $p_t = \frac{1}{N}\sum_{z=1}^{N} p_t(z)$ and $m_t = \frac{1}{N}\sum_{z=1}^{N} m_t(z)$. Why? If we aggregate across all the individual island market clearing conditions we can see that:

$$\begin{aligned} \alpha\left[\frac{1}{N}\sum_{z=1}^{N} p_t(z) - \frac{1}{N}\sum_{z=1}^{N} p_t\right] &= \frac{1}{N}\sum_{z=1}^{N} m_t(z) - \frac{1}{N}\sum_{z=1}^{N} p_t(z) \\ \alpha\left[p_t - \frac{1}{N}Np_t\right] &= m_t - p_t \\ \alpha\left[p_t - p_t\right] &= m_t - p_t \\ 0 &= m_t - p_t \Rightarrow m_t = p_t. \end{aligned}$$

Now, we will make some assumptions about the evolution of $m_t(z)$. Suppose that the log money supply for an island is given by $m_t(z) = m_t + \eta_t(z)$, where $\eta_t(z)$ is an idiosyncratic, mean zero shock. Further, suppose that the aggregate money supply evolves according to $m_t = m_{t-1} + \mu + \xi_t$, where μ denotes a common constant money growth rate and ξ_t denotes a random, mean zero, aggregate money supply shock.

With the above assumptions, we can close the model, and solve for individual and aggregate prices and quantities as functions of the exogenous parameters. Note that the individual island price path becomes:

$$\begin{aligned} p_t(z) &= \frac{m_t(z) + \alpha p_t}{1+\alpha} \\ &= \frac{m_t + \eta_t(z) + \alpha m_t}{1+\alpha} \\ &= \frac{m_{t-1} + \mu + \xi_t + \eta_t(z) + \alpha\left(m_{t-1} + \mu + \xi_t\right)}{1+\alpha} \\ &= m_{t-1} + \mu + \xi_t + \frac{1}{1+\alpha}\eta_t(z) \\ &= m_t + \frac{1}{1+\alpha}\eta_t(z). \end{aligned}$$

This in turn implies that individual island production in equilibrium will be:

$$\begin{aligned} y_t(z) &= \alpha\left[p_t(z) - p_t\right] \\ &= \alpha\left[m_t + \frac{1}{1+\alpha}\eta_t(z) - m_t\right] \\ &= \frac{\alpha}{1+\alpha}\eta_t(z). \end{aligned}$$

From the aggregate market clearing condition, we know that $y_t = m_t - p_t = 0$. Or more precisely, we can see that:

$$\begin{aligned} y_t &= \frac{1}{N}\sum_{z=1}^{N} y_t(z) \\ &= \frac{1}{N}\sum_{z=1}^{N} \frac{\alpha}{1+\alpha}\eta_t(z) \\ &= \frac{\alpha}{1+\alpha}\left[\frac{1}{N}\sum_{z=1}^{N}\eta_t(z)\right] \\ &= \frac{\alpha}{1+\alpha}\left[\bar{\eta}_t(z)\right]. \end{aligned}$$

Assuming that all islands are atomistic (so small that they do not move the aggregate market individually), we effectively have $N \to \infty$. By a law of large numbers, we know that:

$$\frac{1}{N}\sum_{z=1}^{N}\eta_t(z) \to E(\eta_t(z)) = 0, \text{ as } N \to \infty.$$

Recall that we assumed that $\eta_t(z)$ was mean zero. Hence, we can see that $y_t = 0$.

Under perfect information, there is no distortion to the individual island's production decision. Islands respond only to those changes in price which are due to a true change in relative prices, denoted by the individual island shock term $\eta_t(z)$.

2 The Lucas Island Model under Imperfect Information

Suppose that the log price of an individual island's good is given by $p_t(z) = p_t + \lambda_t(z)$, where $\lambda_t(z)$ is an idiosyncratic, mean zero island good relative price shock. To understand this interpretation, note that:

$$\begin{aligned} p_t(z) &= p_t + (p_t(z) - p_t) \\ &= p_t + \lambda_t(z). \end{aligned}$$

Hence, the island's good price differs from the aggregate price only by an idiosyncratic relative price shock term (assuming independence of p_t and $\lambda_t(z)$). Further, suppose that an individual island's production decision depends not on the actual aggregate price level, but rather on their expectation of the aggregate price level (they do not observe the actual aggregate price level exactly). Individual islands behave in a certainty-equivalent manner. Let their expectation of the aggregate price level given individual island information set $I_t(z)$ be denoted $E(p_t|I_t(z)) = \bar{p}_t$. Let the actual aggregate price level differ from this expectation by some independent, mean zero shock term, u_t. Essentially, expectations are rational. We have that $p_t = E(p_t|I_t(z)) + u_t = \bar{p}_t + u_t$. Then, the island supply function is:

$$\begin{aligned} y_t(z) &= \alpha[p_t(z) - \bar{p}_t] \\ &= \alpha[p_t + \lambda_t(z) - \bar{p}_t] \\ &= \alpha[\bar{p}_t + u_t + \lambda_t(z) - \bar{p}_t] \\ &= \alpha[u_t + \lambda_t(z)]. \end{aligned}$$

All that individual islands observe at a point in time is $[u_t + \lambda_t(z)]$; they are unable to differentiate between the aggregate level price shock term and the idiosyncratic relative price shock. Their optimal production choice should only respond to the relative price change, $\lambda_t(z)$.

The problem individual islands face is trying to distinguish between the idiosyncratic shock $\lambda_t(z)$ and the aggregate shock u_t. Assume that individual islands have historical data relating the realizations of $\lambda_t(z)$ and $[u_t + \lambda_t(z)]$ (retrospective data on $\lambda_t(z)$ exists). Then, assume that they can calculate a linear predictor of $\lambda_t(z)$ by regressing $\lambda_t(z)$ on $[u_t + \lambda_t(z)]$.[1] Hence, we have that $\hat{\lambda}_t(z) = \hat{\theta}(z)[u_t + \lambda_t(z)]$, where $\hat{\theta}(z) = \frac{Cov(\lambda_t(z),[u_t+\lambda_t(z)])}{Var([u_t+\lambda_t(z)])} = \frac{Var(\lambda_t(z))}{Var(u_t)+Var(\lambda_t(z))} \in (0,1)$. With this estimator of $E(\lambda_t(z)|I_t(z), p_t(z))$in hand, we can see that:

$$\begin{aligned} &E[p_t(z) - p_t|I_t(z), p_t(z)] = E(\lambda_t(z)|I_t(z), p_t(z)) \\ &\Rightarrow p_t(z) - E[p_t|I_t(z), p_t(z)] = \hat{\theta}(z)[u_t + \lambda_t(z)] \\ &\Rightarrow E[p_t|I_t(z), p_t(z)] = p_t(z) - \hat{\theta}(z)[u_t + \lambda_t(z)] \\ &\Rightarrow E[p_t|I_t(z), p_t(z)] = p_t(z) - \hat{\theta}(z)[p_t(z) - \bar{p}_t] \\ &\Rightarrow E[p_t|I_t(z), p_t(z)] = \left(1 - \hat{\theta}(z)\right)p_t(z) + \hat{\theta}(z)\bar{p}_t \end{aligned}$$

[1] In fact, if we assume that the monetary shock u_t and the idiosyncratic relative price shock $\lambda_t(z)$ are normally distributed, then their sum $[u_t + \lambda_t(z)]$ is also normally distributed. Given that $\lambda_t(z)$ and $[u_t + \lambda_t(z)]$ are jointly normally distributed, the expectation of one is a linear function of the other. Hence, our use of a linear predictor for $\lambda_t(z)$ is justified.

This equation tells us how information on $p_t(z)$ updates our expectation of p_t. Essentially, we have added movements in $p_t(z)$ to our conditioning information set in forming our expectation of p_t. Recall that $\bar{p}_t = E(p_t|I_t(z))$; it is the island's expectation of p_t in the absence of information on $p_t(z)$.

We can employ this formula to update the individual island supply function, so that islands are taking full account of all available information. We now have:

$$\begin{aligned} y_t(z) &= \alpha\left[p_t(z) - E\left[p_t|I_t(z), p_t(z)\right]\right] \\ &= \alpha\left[p_t(z) - \left(1-\hat{\theta}(z)\right)p_t(z) - \hat{\theta}(z)\bar{p}_t\right] \\ &= \alpha\left[\hat{\theta}(z)p_t(z) - \hat{\theta}(z)\bar{p}_t\right] \\ &= \alpha\hat{\theta}(z)\left[p_t(z) - \bar{p}_t\right]. \end{aligned}$$

Notice how this island supply function differs slightly from the original formulation, through the inclusion of $\hat{\theta}(z)$. This is the Lucas supply function.

To complete the model, we must invoke the demand side. Assume that the demand side is identical to the perfect information case:

$$\begin{aligned} y_t^d(z) &= m_t(z) - p_t(z) \\ m_t(z) &= m_t + \eta_t(z) \\ m_t &= m_{t-1} + \mu + \xi_t, \end{aligned}$$

where the variables are defined as in the perfect information case.

Market equilibrium requires that:

$$\begin{aligned} y_t(z) &= y_t^d(z) \\ \alpha\hat{\theta}(z)\left[p_t(z) - \bar{p}_t\right] &= m_t(z) - p_t(z) \\ \alpha\hat{\theta}(z)\left[p_t(z) - \bar{p}_t\right] &= m_{t-1} + \mu + \xi_t + \eta_t(z) - p_t(z). \end{aligned}$$

Aggregating this up to the economy-level, we have that:

$$\begin{aligned} \alpha\hat{\theta}(z)\left[p_t - \bar{p}_t\right] &= m_{t-1} + \mu + \xi_t - p_t \\ \left[1 + \alpha\hat{\theta}(z)\right]p_t &= m_{t-1} + \mu + \xi_t + \alpha\hat{\theta}(z)\bar{p}_t \\ \left[1 + \alpha\hat{\theta}(z)\right]E\left(p_t|I_t(z)\right) &= E\left(m_{t-1}|I_t(z)\right) + E\left(\mu|I_t(z)\right) + E\left(\xi_t|I_t(z)\right) + \alpha\hat{\theta}(z)E\left(\bar{p}_t|I_t(z)\right) \\ \left[1 + \alpha\hat{\theta}(z)\right]\bar{p}_t &= m_{t-1} + \mu + \alpha\hat{\theta}(z)\bar{p}_t \\ \bar{p}_t &= m_{t-1} + \mu. \end{aligned}$$

We can then take this result and apply it to the individual island production decision, since islands have identical information. We then have that:

$$\begin{aligned} \alpha\hat{\theta}(z)\left[p_t(z) - m_{t-1} + \mu\right] &= m_{t-1} + \mu + \xi_t + \eta_t(z) - p_t(z) \\ \left[1 + \alpha\hat{\theta}(z)\right]p_t(z) &= \left[1 + \alpha\hat{\theta}(z)\right]\left[m_{t-1} + \mu\right] + \xi_t + \eta_t(z) \\ p_t(z) &= m_{t-1} + \mu + \frac{\xi_t + \eta_t(z)}{\left[1 + \alpha\hat{\theta}(z)\right]}. \end{aligned}$$

Aggregating this up to the economy level again, we have that:

$$\begin{aligned} p_t &= m_{t-1} + \mu + \frac{\xi_t}{\left[1 + \alpha\hat{\theta}(z)\right]}, \text{ since } \bar{\eta}_t = 0. \\ &\Rightarrow E\left(p_t|I_t(z)\right) = m_{t-1} + \mu = \bar{p}_t, \text{ as above.} \end{aligned}$$

These expectations are model-consistent. Hence, the individual island production function in terms of the exogenous variables is:

$$\begin{aligned} y_t(z) &= \alpha\hat{\theta}\left[m_{t-1} + \mu + \frac{\xi_t + \eta_t(z)}{\left[1 + \alpha\hat{\theta}(z)\right]} - m_{t-1} - \mu\right] \\ &= \left(\frac{\alpha\hat{\theta}}{1 + \alpha\hat{\theta}(z)}\right)(\xi_t + \eta_t(z)) \\ &\text{and aggregate output is given by} \\ y_t &= \left(\frac{\alpha\hat{\theta}}{1 + \alpha\hat{\theta}(z)}\right)\xi_t. \end{aligned}$$

Under imperfect information, we can see that shocks to the aggregate money supply (ξ_t) generate inefficiencies (island production responds to aggregate price changes and not just to relative price changes) and an upward-sloping aggregate supply curve (y_t and p_t are positively related through the money shocks).