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# On measuring deprivation and living standards of societies in a multi-attribute framework

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When measuring a society's deprivation in a multi-attribute framework, researchers often resort to what we call a 'column-first two-stage procedure'. Under such procedures one first determines the society's deprivation for each attribute separately by aggregating the individuals' deprivation levels in terms of that attribute, and then assesses the society's overall deprivation by aggregating the society's deprivation levels for different attributes. In this paper, we argue that all such procedures are seriously flawed insofar as none of them can satisfy simultaneously three highly appealing properties: (i) anonymity, which requires that the individuals be treated symmetrically; (ii) non-invariance, which reflects the sensitivity of the society's overall deprivation to certain switches of deprivation levels between individuals; and (iii) positive responsiveness, which requires that the society's overall deprivation must increase if the society's deprivation for some attribute increases without any decrease in the society's deprivation for any attribute.

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## 1. Introduction

Over the last decade or so, economists have increasingly measured the deprivation and living standards of societies in a multi-attribute framework. There has been a growing realization that, while income-based measures of deprivation and living standards are useful, they have important conceptual limitations (see Sen, 1985, 1987, for foundational statements of this criticism), and, to overcome these limitations, one needs to think directly in terms of other valuable attributes such as outcomes in (or the specific means to achieve) health, education, housing, personal security, etc. Such thinking underlies, for instance, the measurement exercises undertaken in the well known Human Development Reports of the United Nations Development Programme (UNDP).

This paper studies the implications of certain highly attractive properties that one may like to impose on measures of deprivation in a multidimensional framework. In particular, one of our results shows that a large class of measures of social deprivation, which we call the 'column-first two-stage procedures for aggregating deprivation matrices' (or, 'column-first two-stage procedures' in short) and which includes the UNDP's influential Human Poverty Index as a special case, fails to satisfy three very plausible properties. To articulate the significance of this result, we first describe intuitively what the class of column-first two-stage procedures encompasses and what the properties under consideration are.

We start with an example to describe intuitively column-first two-stage procedures as distinct from what we call row-first two-stage procedures. For the purpose of this example, consider a society of two individuals, 1 and 2, and exactly two attributes, health and education. Assume that the deprivation of each individual in terms of each attribute is measured on a scale from 0 to 1. Let us also assume that the problem of measuring overall deprivation of the society is one of finding a real-valued (ordinal) index for the society's deprivation, given the information about each individual's deprivation in terms of each attribute. Suppose the information about the individuals' deprivation levels in terms of the two attributes is given by the following table:

**Table 1**

| | Health | Education |
|---|---|---|
| 1 | 0.3 | 0.1 |
| 2 | 0.2 | 0.5 |

where 0.3 indicates 1's health deprivation, 0.5 indicates 2's educational deprivation, and so on. A column-first two-stage procedure starts with the columns of numbers first. Thus, one takes the health deprivation levels (0.3 for 1 and 0.2 for 2) in the first column and aggregates them to arrive at an index of health deprivation of the society. Similarly, one arrives at the index of educational deprivation of the society from the second column of numbers (0.1 for 1 and 0.5 for 2). In the next stage one aggregates the society's health deprivation index and the society's educational deprivation index to arrive at the society's overall deprivation index. The sequence is very different under row-first two-stage procedures. Here one first looks at the rows of numbers. Thus, one would take individual 1's deprivation indices (0.3 for health and 0.1 for education) figuring in the first row and aggregate them to get an index of 1's overall deprivation. Similarly one would aggregate the numbers in the second row to get an index of

2's overall deprivation. In the next stage, one would aggregate the two individuals' overall deprivation indices already derived in the first stage to arrive at the index of overall deprivation for the society.

One of our main results (Proposition 3) demonstrates that no column-first two-stage procedure can satisfy simultaneously three highly plausible properties, namely, anonymity, non-invariance, and positive responsiveness. Anonymity requires that, if, other things remaining the same, we interchange the deprivation levels of two individuals with respect to each of the attributes or dimensions under consideration, then the new situation should have the same level of overall social deprivation as the initial situation.

This, of course, is the exact counterpart of the property of anonymity used in the measurement of deprivation on the basis of income alone; in each case, the property is highly appealing if one wishes to abstract from issues involving discrimination based on race, gender, caste, etc. The second property, non-invariance, which is concerned with how certain switches of deprivation levels between two individuals may affect overall deprivations of the society, is also highly plausible, especially when at least two attributes are 'substitutes' of each other. Anonymity and non-invariance are properties defined for all measures of overall social deprivation including column-first two-stage procedures. Our third property, positive responsiveness, is defined exclusively for column-first two-stage procedures. It is a compelling property of column-first two-stage procedures: it requires that overall social deprivation must increase if, the individual deprivations in terms of the different attributes change in such a way that, at the end of the first stage of the column-first two-stage procedure, social deprivation in terms of at least one attribute increases and social deprivation in terms of no attribute decreases.

Our result that no column-first procedure can simultaneously satisfy anonymity, non-invariance, and positive responsiveness is highly disturbing. Column-first two-stage procedures have the attraction of being practical, especially if data concerning the levels of deprivations experienced by members of the group are available from independent sources for different attributes and we do not have the joint distribution for all the attributes. It is not, therefore, surprising that column-first two-stage procedures have been widely used (see, among others, Morris, 1979, and the UNDP's well known Human Poverty Index). Our result, however, shows that all column-first two-stage procedures are seriously flawed: they can satisfy anonymity and non-invariance only at the cost of violating positive-responsiveness. Thus, while limitations of data may force us to use column-first procedures, we can use these procedures only with the uncomfortable feeling that something may be seriously amiss somewhere in our exercise. The only escape from this problem lies in the coordinated collection of more detailed data with information about joint distributions for attributes, so that we can use measures of social deprivation, which can accommodate more adequately our basic intuition about deprivation.

## 2. Notation and definitions

Let $N = \{1, \ldots, n\}$ be a given finite set of individuals ($n \geqslant 2$) and let $F = \{f_1, \ldots, f_m\}$ be a given finite set of attributes ($m \geqslant 2$). Let $M = \{1, \ldots, m\}$. For every $j \in M$, let $R^j$ be a non-empty set of real numbers; we assume that $\#R^j \geqslant 2$. Given our focus on the measurement of deprivation, we shall assume that, for every $j \in M$, $0 \in R^j \subseteq [0, 1]$ and we shall interpret $R^j$ as the different levels of deprivation in terms of attribute $f_j$ that an individual may possibly have, with 0 indicating the absence of any deprivation in terms of the attribute under consideration. For our purpose, it is enough to assume that the numbers in $R^j$ have an ordinal significance so that, if $\alpha$, $\beta \in R^j$ and $\alpha > \beta$, then $\alpha$ represents a higher level of deprivation in terms of attribute $f_j$ than $\beta$, but we do not rule out the possibility of these numbers having cardinal significance. The economy achieved by assuming no more than ordinal significance for the different possible levels of deprivation for any attribute is important insofar as deprivation may not be cardinally measurable in the case of many attributes such as health.

Let $(a_{ij})_{n \times m}$ be an $n \times m$ matrix of real numbers such that, for all $i \in N$ and all $j \in M$, $a_{ij} \in R^j$. For all $i \in N$ and all $j \in M$, $a_{ij}$ will be interpreted as the level of individual $i$'s deprivation in terms of attribute $f_j$. We shall refer to the matrix $(a_{ij})_{n \times m}$ as a deprivation matrix. The $n \times m$ matrices $(a_{ij})_{n \times m}$, $(a'_{ij})_{n \times m}, \ldots,$ will be denoted by $A$, $A'$, etc. The class of all such $n \times m$ matrices will be denoted by $V$. For every $A = (a_{ij})_{n \times m} \in V$ and every individual $p$, let $a_{p\bullet}$ denote the row vector $(a_{p1}, \ldots, a_{pm})$, indicating individual $p$'s deprivation levels in terms of the $m$ attributes. Likewise, for every $A = (a_{ij})_{n \times m} \in V$ and every attribute $f_j$, let $a_{\bullet j}$ denote the column vector $(a_{1j}, \ldots, a_{nj})$, indicating each individual's deprivation level in terms of the attribute $f_j$. Let $A$, $A' \in V$, $i \in N$, and $j \in M$. We say that $a_{i\bullet}$ and $a'_{i\bullet}$ are $j$–variants if and only if $a_{ij} \neq a'_{ij}$ and $a_{ik} = a'_{ik}$ for all $k \in M - \{j\}$, that is, if and only if $a_{i\bullet}$ and $a'_{i\bullet}$ are identical except that $a_{ij} \neq a'_{ij}$.

Let $\succcurlyeq$ be a reflexive (but not necessarily transitive or connected) binary relation over $V$. We shall call such a binary relation an overall group deprivation ranking (OGDR). For all $A$, $A' \in V$, $A \succcurlyeq A'$ denotes that the overall deprivation of the group $N$ in the social situation given by $A$ is deemed at least as high as the overall group deprivation in the social situation described by $A'$. For all $A$, $A' \in V$, $[A \succ A'$ iff $(A \succcurlyeq A'$ and not$(A' \succcurlyeq A))]$ and $[A \sim A'$ iff $(A \succcurlyeq A'$ and $A' \succcurlyeq A)]$. $A \succ A'$ indicates that the overall group deprivation is deemed strictly greater in the social situation $A$ than in the social situation $A'$, and $A \sim A'$ indicates that the overall group deprivation is deemed identical in the two social situations.

In much of the literature on multi-dimensional deprivation, the group deprivation measure specifies exactly one real number for each deprivation matrix. Of course, any such group deprivation measure induces an ordering over $V$. We have, however, chosen a more general framework in which the binary relation $\succcurlyeq$ over $V$ rather than the real numbers attached to different deprivation matrices is the primitive concept. Our negative results proved for $\succcurlyeq$ will encompass, *a fortiori*, corresponding results for the case where, for every

$A \in V$, we have a real number representing the level of overall group deprivation corresponding to $A$.

## 3. Anonymity and non-invariance

In this section, we shall consider some appealing properties which may be imposed on an OGDR, $\succcurlyeq$, and which are related to the invariance of the OGDR to permutations of different kinds.

### 3.1 Anonymity

Consider two deprivation matrices $A$ and $B$ such that $A$ and $B$ are identical except that, for some two individuals $s$ and $t$, $s$'s deprivation levels under $A$ are identical to $t$'s deprivation levels under $B$, and $t$'s deprivation levels under $A$ are identical to $s$'s deprivation levels under $B$; that is, $a_{s\bullet} = b_{t\bullet}$, $a_{t\bullet} = b_{s\bullet}$, and $a_{k\bullet} = b_{k\bullet}$ for all $k \neq s, t$. If one believes that all individuals should be treated symmetrically, then one would require that $A$ and $B$, as specified above, should be associated with the same overall group deprivation level. Formally, this requirement is captured by the following:

*Anonymity (A)* $\succcurlyeq$ satisfies anonymity (A) if and only if, for all $A, B \in V$ and all $s, t \in N$, if ($a_{i\bullet} = b_{i\bullet}$ for all $i \in N - \{s, t\}$, $a_{s\bullet} = b_{t\bullet}$, and $a_{t\bullet} = b_{s\bullet}$ ), then $A \sim B$.

Anonymity essentially requires that information about individuals, such as their names, not already captured in the deprivation matrix should not play any role in constructing an OGDR and an OGDR should be neutral with respect to such information. Having said this, we note that anonymity would not be a reasonable property to postulate for $\succcurlyeq$ if we want to build into our measure any extra concern (beyond that already taken into account in the assessment of their individual deprivation levels, registered in the deprivation matrix) about the deprivation of specific subgroups in $N$, such as women, ethnic minorities, people belonging to a lower caste, etc. In the absence of any such special concern about any subgroup, however, the property seems to have a powerful appeal and has been used extensively in the literature on poverty and deprivation.

### 3.2 Non-invariance

We now present our next property, which we call 'non-invariance'. We first introduce the formal definition of non-invariance and interpret it. Next, we consider several justifications for non-invariance.

*Non-invariance (NIN)* $\succcurlyeq$ satisfies non-invariance (NIN) if and only if there exist $A, B \in V$, $s, t \in N$, and $k \in M$, such that ($a_{\bullet j} = b_{\bullet j}$ for all $j \in M - \{k\}$), ($a_{ik} = b_{ik}$ for all $i \in N - \{s, t\}$), ($a_{sk} = b_{tk}$), ($a_{tk} = b_{sk}$), and not($A \sim B$).

Formally, NIN requires the existence of some deprivation matrix $A$, some individuals $s$ and $t$ and some attribute $f_k$, such that, if, starting with $A$, we interchange the deprivation levels of $s$ and $t$ with respect to attribute $f_k$ in $A$, keeping everything else in $A$ unchanged, then the new deprivation matrix will have a different level of overall group deprivation as compared to $A$.

To elaborate the intuition underlying NIN, we consider the following example.

*Example 1* Consider a society with two individuals, 1 and 2, and two attributes, $f_1$ and $f_2$. Consider first a situation represented by the following deprivation matrix $X$:

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Note that, in $X$, individual 1 is deprived in terms of each attribute while individual 2 is not deprived in terms of any attributes, so that individual 1 may be said to be unambiguously more deprived than individual 2. Now assume that, as far as attributes $f_1$ is concerned, there are no changes in the deprivation of either individual, while, for attribute $f_2$, there is a switch of the two individual's deprivation levels. The new situation is depicted by the matrix $Y$ below:

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

One's initial intuition may lead one to claim that the situation under $X$ should be deemed to involve greater overall group deprivation than the situation under $Y$. The normative rationale for such a claim may be thought to be similar to that underpinning the Pigou-Dalton transfer principle in the literature of income inequality or (more immediately) the 'prioritarian' principle described in the philosophical literature (see, e.g., Parfit, 1997; see also Anand, 1983, Appendix E, for a demonstration that in the single-attribute context this principle is entailed by an 'egalitarian' social welfare function): the 'transfer' of an amount of deprivation of an attribute from a more deprived individual to a less deprived individual should be deemed to reduce the overall group deprivation level. This intuition, however, needs careful scrutiny. First, consider the specific case where $f_1$ denotes nourishment in general and $f_2$ denotes either the absence of morbidity or protection from the elements. Intuitively, it seems plausible to say that, given the level of nourishment, the damage caused to a person by a given increase in morbidity (or by a given increase in exposure to harsh cold) will be greater if the person's nourishment is fixed at a lower level rather than at a higher level. In this case, $f_1$ and $f_2$ can be said to be substitutes. In the general case where we have $m$ ($m \geqslant 2$) attributes, we shall say that two attributes, $f_k$ and $f_{k'}$ are substitutes (resp. complements) if the harm caused by a given decrease in the amount of $f_{k'}$, when the amount of every other attribute is held fixed, will be smaller (resp. larger) if the level at which the amount of $f_k$ is held fixed is higher rather than lower, and we say that $f_k$ and $f_{k'}$ are independent of each other if the harm caused by a given decrease in the amount of $f_{k'}$, when the amount of every other attribute is held fixed, does not depend on the level at which the amount of $f_k$ is fixed. If $f_1$ and $f_2$ are substitutes, we can discern two distinct implications of the switch from $X$ to $Y$: first, the initially more deprived person gains at the expense of the initially less deprived person and, second, the 'real' loss of the initially less deprived person is less than the 'real' gain of the initially more deprived person. Since both these effects work in the same direction so far as our intuition about the implications, for overall social deprivation, of the switch from $X$ to $Y$ is concerned, it seems intuitively compelling to say that, when $f_1$ and $f_2$ are substitutes, the switch from $X$ to $Y$ decreases overall social deprivation. Bourguignon and Chakravarty (2009) make similar observations. Noting that a change from $X$ to $Y$ involves simply an interchange of $f_2$-deprivations of individuals 1 and 2, it is clear that, when $f_1$ and $f_2$ are substitutes, NIN must hold. If $f_1$ and $f_2$ are complements, it can be easily seen that again there will be two analogous effects when we move from $X$ to $Y$, but, in this case, the two effects will work in opposite directions so far as our intuition about overall social deprivation is concerned, and, hence, our intuition about the impact that the move from $X$ to $Y$ has on social deprivation will remain ambiguous. If $f_1$ and $f_2$ are independent of each other, then the change from $X$ to $Y$ will mean that the initially more deprived person will gain at the expense of the initially less deprived person and the initially more deprived person's real gain is of the same magnitude as the initially less deprived person's real loss. While one's intuition here is less clear-cut than our intuition in the case where $f_1$ and $f_2$ are substitutes, many people may still feel that a move from $X$ to $Y$ in this case decreases overall deprivation of the society.

In general, when we have $m$ $(m \geqslant 2)$ attributes and $f_k$ and $f_{k'}$ are two attributes which are substitutes in the sense explained above, the following would seem to be intuitively very plausible:

(i) for all $A$, $B \in V$ and for all $s$, $t \in N$, if $[(a_{i\bullet} = b_{i\bullet}$ for all $i \in N - \{s, t\})$ and (for all $j \in M - \{k, k'\}$, $a_{sj} = b_{sj} = a_{tj} = b_{tj}$; $a_{sk'} = b_{sk'} > a_{tk'} = b_{tk'}$; and $a_{sk} = b_{tk} > a_{tk} = b_{sk})]$, then $A \succ B$.

(i) is, of course, much stronger than NIN. Thus, if at least two attributes are substitutes, then NIN would seem to be an intuitively compelling property, though, of course, NIN can hold even when no two attributes are substitutes of each other. Indeed, when the two attributes are complements, the class of multi-dimensional deprivation indices suggested by Bourguignon and Chakravarty (1999) satisfies NIN.

NIN is closely related to the notions of non-decreasing poverty under correlation increasing switches (NDCIS) and non-increasing poverty under correlation increasing switches (NICIS) formally introduced in Bourguignon and Chakravarty (1999) based on a similar concept in Atkinson and Bourguignon (1982) for multivariate distributions of economic status (see Bourguignon and Chakravarty, 2003, and Tsui, 2002, for further discussions of the notions of NDCIS and NICIS). In our Example 1 involving deprivation matrices $X$ and $Y$, the movement from the deprivation matrix $Y$ to the deprivation matrix $X$ is a switch that increases the correlation of the attributes within the population. Then, the notion of non-increasing deprivation under correlation increasing switch postulates that overall group deprivation cannot increase with such correlation increasing switches, and the notion of non-decreasing deprivation under correlation increasing switch stipulates that overall group deprivation cannot decrease with such switches. Our notion of non-invariance is logically independent of the notions of non-decreasing and non-increasing poverty under correlation increasing switches as the latter two notions allow $X \sim Y$ to hold for $X$ and $Y$ specified in Example 1 while NIN requires not($X \sim Y$) to hold. It is however clear that, for $\succcurlyeq$ to violate NIN, $\succcurlyeq$ must satisfy an extremely restrictive condition, namely that no correlation increasing (or decreasing) switch of deprivation in terms of any attribute between any two individuals must lead to any change in overall social deprivation.

## 4. Column-first two-stage procedures for aggregating deprivation matrices

We now consider a class of widely used procedures for deriving a binary relation $\succcurlyeq$ over $V$; we shall call them column-first two-stage procedures. Before introducing the formal definition of these procedures, it may be helpful to consider a simple example that illustrates their underlying intuition.

*Example 2* Consider the case where $N = \{1, 2\}$ and $F = \{f_1, f_2\}$. A column-first two-stage procedure for ranking alternative deprivation matrices, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $A' = \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}, \ldots$, proceeds in two stages. First, for each attribute $f_j$ ($j = 1, 2$), it specifies a function $h_j$ for aggregating the deprivations of the two individuals in terms of attribute $f_j$ to arrive at a measure of the society's deprivation in terms of that attribute. Thus, given the deprivation matrix, $A$, the function $h_1$ would aggregate $a_{\bullet 1} = (a_{11}, a_{21})$ to arrive at a real number $\beta_1$, an index of the society's deprivation in terms of $f_1$ in the situation represented by $A$; and, similarly, $h_2$ will aggregate $a_{\bullet 2} = (a_{12}, a_{22})$ to yield a real number $\beta_2$. Thus corresponding to deprivation matrices $A, A', \ldots,$ we would obtain vectors $(\beta_1, \beta_2), (\beta'_1, \beta'_2), \ldots$. In the second stage, those different vectors, $(\beta_1, \beta_2), (\beta'_1, \beta'_2), \ldots$, obtained in the first stage are ranked, and the ranking of these vectors induces a ranking, $\succcurlyeq$, of the matrices $A, A', \ldots$ in terms of overall social deprivation.

In general, a column-first two-stage procedure for deriving a ranking $\succcurlyeq$ over $V$ is as follows. In the first stage, we use $m$ functions $h_j : (R^j)^n \to R$ ($j \in M$) so as to arrive, for each deprivation matrix $A$, at an $m$-tuple of real numbers: $(h_1(a_{\bullet 1}), \ldots, h_m(a_{\bullet m}))$. For every $j \in M$, $h_j(a_{\bullet j})$ can be thought of as the society's deprivation in terms of attribute $j$. Thus, the information in the deprivation matrix $A$ is compressed into a vector $(h_1(a_{\bullet 1}), \ldots, h_m(a_{\bullet m}))$ of real numbers. For each $j \in M$, let $H_j = \{\beta : \beta = h_j(a_{\bullet j})$ for some $A \in V\}$. Let $H = H_1 \times \ldots \times H_m$. In the second stage of the procedure we rank the $m$-vectors $(h_1(a_{\bullet 1}), \ldots, h_m(a_{\bullet m}))$, $(h_1(a'_{\bullet 1}), \ldots, h_m(a'_{\bullet m}))$, etc. corresponding to $A, A', \ldots \in V$. Letting $\succcurlyeq^{cr}$ denote this 'intermediate' ranking over $H$, the ranking of $\succcurlyeq$ over $V$ is induced by the following rule: for all $A$, $A' \in V$, $A \succcurlyeq A'$ if and only if $(h_1(a_{\bullet 1}), \ldots, h_m(a_{\bullet m})) \succeq^{cr} \left(h_1(a'_{\bullet 1}), \ldots, h_m(a'_{\bullet m})\right)$.

We can now provide a formal definition of the notion of an OGDR being derivable through a given column-first two-stage procedure.

Let $\succcurlyeq$ be a given reflexive and transitive binary relation defined over $V$. For every $j \in M$, let $h_j$ be a function from $(R^j)^n$ to $R$, and let $\succcurlyeq^{cr}$ be a reflexive and transitive relation defined over $H$. We say that $\succcurlyeq$ can be derived through the column-first two-stage procedure based on $(h_1, \ldots, h_m; \succcurlyeq^{cr})$ if and only if for all $A$, $A' \in V$, $[A \succcurlyeq A'$ if and only if $(h_1(a_{\bullet 1}), \ldots, h_m(a_{\bullet m})) \succeq^{cr} \left(h_1\left(a'_{\bullet 1}\right), \ldots, h_m\left(a'_{\bullet m}\right)\right)]$.

## 5. Implications of anonymity and non-invariance for group deprivation rankings in general

We first discuss some implications of anonymity and non-invariance for OGDRs. For this purpose, we consider the following property.

*Separability (S)* $\succcurlyeq$ satisfies separability (S) if and only if, for all $A$, $B$, $A'$, $B' \in V$, all $i \in N$ and all $j \in M$, if $[a_{pk} = a'_{pk}$ and $b_{pk} = b'_{pk}$ for all $pk \neq ij]$, and $[a_{ij} = b_{ij}$, and $a'_{ij} = b'_{ij}]$, then $A \succcurlyeq B \Leftrightarrow A' \succcurlyeq B'$.

Suppose we start with two deprivation matrices $A$ and $B$, such that some individual, $i$, has the same level of deprivation in terms of some attribute $j$ in both $A$ and $B$. Now suppose we derive deprivation matrices $A'$ and $B'$ from $A$ and $B$, respectively, by replacing in $A$ and $B$ the identical levels of $i$'s deprivation in terms of attribute $j$ by some other identical levels of $i$'s deprivation in terms of attribute $j$, leaving intact everything else in $A$ and $B$. Then separability requires that $A'$ and $B'$ must be ranked in exactly the same way as $A$ and $B$ are ranked. S has some of the intuitive flavor of the condition of sub-group consistency, which Tsui (2002) introduced earlier. While S is not as compelling a property of $\succcurlyeq$ as A or NIN, it seems to be a feature of many convenient and plausible group decision measures (see Example 3 (ii) below). For our purpose, a formally weaker version of S will suffice.

*Weak separability (WS)* $\succcurlyeq$ satisfies weak separability (WS) if and only if, for all $A$, $B$, $A'$, $B' \in V$, all $i \in N$ and all $j \in M$, if $[a_{pk} = a'_{pk}$ and $b_{pk} = b'_{pk}$ for all $pk \neq ij]$, $[a_{ij} = b_{ij}$, and $a'_{ij} = b'_{ij}]$, and [there exists $e \in M - \{j\}$ such that, for all $k \in M - \{e\}$, $a_{\bullet k} = b_{\bullet k}]$, then $A \succcurlyeq B \Leftrightarrow A' \succcurlyeq B'$.

It is clear that S implies WS though the converse does not hold. We use the weaker condition WS in the following proposition. The proof of the proposition can be found in the Appendix.

*Proposition 1* There does not exist any $\succcurlyeq$ which simultaneously satisfies anonymity, non-invariance, and weak separability.

Thus, every OGDR satisfying anonymity and non-invariance must violate weak separability and hence separability. The failure of $\succcurlyeq$ to satisfy WS and S is not in itself a major reason for concern. Many OGDRs, however, incorporate separability properties in their structure and all such OGDRs will, therefore, be ruled out by the conjunction of anonymity and non-invariance.

*Example 3* We give examples of OGDRs, which satisfy two of the three properties, A, NIN, and WS, figuring in Proposition 1, but violate the third. In these examples, we shall assume that every attribute is measured along the real interval ]0, 1] and can take any value in this interval.

(i) The following OGDR satisfies A and NIN but violates WS. For all deprivation matrices $A$ and $B$,

$$A \succeq B \text{ iff } \frac{1}{n}\sum_{i=1}^{n}\left[a_{i1}^{\alpha_1}\ a_{i2}^{\alpha_2}\ \ldots\ a_{im}^{\alpha_m}\right] \geqslant \frac{1}{n}\sum_{i=1}^{n}[b_{i1}^{\alpha_1}\ b_{i2}^{\alpha_2}\ \ldots\ b_{im}^{\alpha_m}],$$

where $\alpha_j > 1$ $(j = 1, \ldots, m)$.

(ii) The following OGDR satisfies A and S (and hence WS) but violates NIN. For all deprivation matrices $A$ and $B$,

$$A \succeq B \text{ iff } \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^{\alpha_j} \geqslant \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij}^{\alpha_j}$$

where $\alpha_j > 1$ $(j = 1, \ldots, m)$.

(iii) The following ranking satisfies NIN and S (and hence WS) but violates A. For all deprivation matrices $A$ and $B$,

$$A \succeq B \text{ iff } \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} + \sum_{i=j=1}^{\min\{m,n\}} a_{ij} \geqslant \sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij} + \sum_{i=j=1}^{\min\{m,n\}} b_{ij}.$$

To see that NIN is satisfied, let $m = n = 3$, and start with a matrix $A$ such that $[a_{11} = a_{22} = a_{33} = 1]$ and [for all $a_{ij}$, if $i \neq j$, then $a_{ij} = 0.01$]. Next, construct matrix $B$ by interchanging $a_{12}$ and $a_{22}$, leaving everything else unchanged. It can be checked that $A \succ B$, which satisfies NIN. To see the violation of A, derive $C$ from $A$ by interchanging the second and third rows of $A$ and leaving the first row unchanged; it can be checked that $A \succ C$, which violates A.

## 6. Implications of anonymity and non-invariance for column-first two-stage procedure

We now show that, for OGDRs based on column-first two-stage procedures, anonymity and non-invariance, together, precipitate a disturbing consequence.

Consider a column-first two-stage procedure based on $(h_1, \ldots, h_m; \succcurlyeq^{cr})$. We say that $\succcurlyeq^{cr}$ is positively responsive if and only if for all $A, B \in V$, if $(h_1(a_{\bullet 1}), \ldots, h_m(a_{\bullet m})) > (h_1(b_{\bullet 1}), \ldots, h_m(b_{\bullet m}))$, then $(h_1(a_{\bullet 1}), \ldots, h_m(a_{\bullet m})) \succ^{cr} (h_1(b_{\bullet 1}), \ldots, h_m(b_{\bullet m}))$.

*Proposition 2* Let $\succcurlyeq$ be an OGDR derived through a column-first two-stage procedure based on $(h_1, \ldots, h_m; \succcurlyeq^{cr})$ such that $\succcurlyeq^{cr}$ is positively responsive. Then $\succcurlyeq$ satisfies weak separability.

*Proposition 3* Let $\succcurlyeq$ be an OGDR that is derived through a column-first two-stage procedure based on $(h_1, \ldots, h_m; \succcurlyeq^{cr})$ such that $\succcurlyeq^{cr}$ is positively responsive. Then $\succcurlyeq$ cannot satisfy both anonymity and non-invariance.

*Proof* Proposition 3 follows directly from Propositions 1 and 2. □

## 7. An extension

So far we have focused on the problem of measuring deprivation. Our analysis can, however, be readily extended to the measurement of living standards more generally. To do this, one needs to interpret $R^j$ as the set of different levels of achievements in terms of attribute $f_j$, a higher number in $R^j$ denoting a higher level of achievement with respect to attribute $f_j$. The requirement that $R^j \subseteq [0, 1]$ was not drawn upon in our proofs and can be laid aside. As before, we shall have $n \times m$ matrices where each real number figuring in the matrix indicates the level of achievement of some individual $i \in N$ in terms of some attribute $f_j$. The binary relation $\succcurlyeq$ defined over all such matrices will now be interpreted in terms of living standards: for all standard of living matrices, $A$ and $A'$, $A \succcurlyeq A'$ will now denote that $A$ represents a higher standard of living for the society than $A'$. The formal definitions of all the properties, will remain intact.

It is evident that, given these revised versions of the properties of Section 3, the exact counterparts of propositions in the previous three sections hold in the context of standard of living comparisons. Thus, the force of the negative results remains intact when we switch from the measurement of deprivation to the measurement of living standards. Gajdos and Weymark (2005) note similar negative results in the context of standard of living.

## 8. Conclusion

In this paper, we have discussed the implications of imposing two appealing conditions, anonymity and non-invariance, on measures of social deprivation or living standards in a multi-attribute framework. In particular, we have shown that measures of social deprivation and living standards derived through the widely used column-first two-stage procedures cannot possibly satisfy both these basic properties unless the column-first procedures on which they are based violate the compelling property of positive responsiveness in the second stage of aggregation. On the other hand, it is of interest to note that row-first two-stage procedures do not have this deficiency. An important message of our results is that countries and institutions should collect data on different dimensions of deprivation and well-being for each of the individuals under consideration (see Dutta *et al.*, 2003, for related observations).

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## Appendix

*Proof of Proposition 1* Suppose to the contrary that there exists an OGDR $\succcurlyeq$ satisfying anonymity, non-invariance and weak separability. By non-invariance, there exist $A, B \in V$, $s, t \in N$ and $j \in M$ such that $[a_{\bullet k} = b_{\bullet k}$ for all $k \in M - \{j\}]$

$[a_{sj}=b_{tj},\ a_{tj}=b_{sj}]$, $[a_{ij}=b_{ij}$ for all $i\in N-\{s,t\}]$, and $[\text{not}(A\sim B)]$. Without loss of generality, let $s=1$, $t=2$, and $j=2$. Then

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix}, \quad B=\begin{pmatrix} a_{11} & a_{22} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{12} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix}$$

and not$(A\sim B)$. Consider $A^1, B^1, A^2, B^2, \ldots, A^{m-1}$, $B^{m-1}\in V$ such that: $A^1$ is obtained from $A$ by changing $a_{21}$ to $a_{11}$ while keeping all other entries of $A$ unchanged, and $B^1$ is obtained from $B$ by changing $a_{21}$ to $a_{11}$ while keeping all other entries of $B$ unchanged; $A^2$ is obtained from $A^1$ by changing $a_{23}$ to $a_{13}$ while keeping all other entries of $A^1$ unchanged, and $B^2$ is obtained from $B^1$ by changing $a_{23}$ to $a_{13}$ while keeping all other entries of $B^1$ unchanged; ...; and $A^{m-1}$ is obtained from $A^{m-2}$ by changing $a_{2m}$ to $a_{1m}$ while keeping all other entries of $A^{m-2}$ unchanged, and $B^{m-1}$ is obtained from $B^{m-2}$ by changing $a_{am}$ to $a_{1m}$ while keeping all other entries of $B^{m-2}$ unchanged. Note that, $A^{m-1}$ and $B^{m-1}$ are the following two matrices:

$$A^{m-1}=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{11} & a_{22} & a_{13} & \cdots & a_{1m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix}, \quad B^{m-1}=\begin{pmatrix} a_{11} & a_{22} & a_{13} & \cdots & a_{1m} \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix}$$

By weak separability: from not$(A\sim B)$, we obtain not$(A^1\sim B^1)$, and from not$(A^1\sim B^1)$, we obtain not$(A^2\sim B^2)$, etc., and from not$(A^{m-2}\sim B^{m-2})$, we obtain not$(A^{m-1}\sim B^{m-2})$. Note, however, by anonymity, $A^{m-1}\sim B^{m-1}$, a contradiction. Therefore, there exists no OGDR satisfying anonymity, non-invariance and weak separability. □

*Proof of Proposition 2* Let $\succcurlyeq$ be an OGDR derived through a column-first two-stage procedure based on $(h_1,\ldots,h_m;\ \succcurlyeq^{cr})$ such that $\succcurlyeq^{cr}$ is positively responsive. Consider $A, B, A', B'\in V$, $i\in N$ and $j\in M$, such that $[a_{pk}=a'_{pk}$ and $b_{pk}=b'_{pk}$ for all $pk\neq ij]$, $[a_{ij}=b_{ij}$, and $a'_{ij}=b'_{ij}]$, and [there exists $e\in M-\{i\}$, such that $a_{\bullet k}=b_{\bullet k}$ for all $k\in M-\{e\}$]. It is clear that [for all $k\in M-\{e\}$, $h_k(a_{\bullet k})=h_k(b_{\bullet k})$ and $h_k(a'_{\bullet k})=h_k(b'_{\bullet k})$] and $[h_e(a_{\bullet e})=h_e(a'_{\bullet e})$, and $h_e(b_{\bullet e})=h_e(b'_{\bullet e})]$.

Suppose $h_e(a_{\bullet e})=h_e(b_{\bullet e})$. Then $h_e(a'_{\bullet e})=h_e(b'_{\bullet e})$. In that case, it is clear that, for all $k\in M$, $h_k(a_{\bullet k})=h_k\ (b_{\bullet k})$ and $h_k(a'_{\bullet k})=h_k(b'_{\bullet k})$. Then, by the reflexivity of $\succcurlyeq^{cr}$, $A\sim B$ and $A'\sim B'$.

Suppose $h_e(a_{\bullet e}) > h_e(b_{\bullet e})$. Then $h_e(a'_{\bullet e}) > h_e(b'_{\bullet e})$. In that case, $(h_1(a_{\bullet 1}), \ldots, h_m(a_{\bullet m})) > (h_1(b_{\bullet 1}), \ldots, h_m(b_{\bullet m}))$ and $\big(h_1(a'_{\bullet 1}), \ldots, h_m(a'_{\bullet m})\big) > \big(h_1(b'_{\bullet 1}), \ldots, h_m(b'_{\bullet m})\big)$. Then, by positive responsiveness of $\succcurlyeq^{cr}$, $A \succ B$ and $A' \succ B'$.

Similarly, it can be shown that, if $h_e(a_{\bullet e}) < h_e(b_{\bullet e})$, then $B \succ A$ and $B' \succ A'$.

Thus, $A'$ and $B'$ are ranked in exactly the same way as $A$ and $B$. This shows that $\succcurlyeq$ satisfies weak separability. $\square$