Thin Ice Growth

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The thickening of ice covers has traditionally been calculated using a method in which the thickness is assumed proportional to the square root of the accumulated freezing degree days. Particularly for newly formed thin ice this method overpredicts the ice thickness. Consideration of the thermal resistance between the top of the ice and the atmosphere results in a method which predicts linear growth with time for thin ice and transitions to the $t^{1/2}$ growth at large thicknesses. The method proposed here is not new but seems to be rarely used even though it requires selection of only one coefficient. Data from several sources including initial river ice growth, sea ice growth, and sludge freezing are used to validate the method and bracket the coefficient.

INTRODUCTION

The most commonly used method for predicting the thickness of floating ice sheets is the so-called "square root of freezing degree days" method. In this method the difference between the daily average air temperature and the freezing point of water is multiplied by time (days) since initial ice formation, the square root taken, and the result multiplied by a coefficient to obtain the predicted ice thickness. This method seems to work reasonably well for thicknesses over about 10 cm. For thicknesses less than about 10 cm it will be shown below that the method overpredicts the ice thickness. A method nearly as simple is shown to give good results at all thicknesses. The additional consideration is the effect of the thermal resistance between the top of the ice and the atmosphere above the ice. For some cases use of the traditional method can give very erroneous results.

The purpose of this paper is to more widely publicize the more correct method since, even though it has been known for many years, is often not used in cases where the difference between the two methods is important.

The method is not meant to substitute for analyses of ice growth that require considerably greater data needs to evaluate the components of the energy budget necessary to perform detailed assessment of ice growth as, for example, described by *Kuroda* [1985] or *de Bruin and Wessels* [1988], or for cases where one component of the energy budget is crucial to the particular ice growth process of interest such as the analysis of deterioration under the influence of solar radiation [*Ashton*, 1985].

TRADITIONAL METHOD

The most commonly used method for predicting the thickness of ice is based on a very simplified solution of the so-called Stefan problem. That is, given an ice sheet growing into the melt below, with a fixed top surface temperature, what is the relation between the thickness and time? The simplified solution is obtained by expressing the heat flux through the ice in the form

$$Q_i = -k(T_m - T_s)/h \qquad (1)$$

where Q_i is the heat flux through the ice, k is the thermal conductivity of the ice, h is the thickness, T_m is the temper-

ature at the ice water interface ($=0°C$), and T_s is the temperature of the top surface. At the bottom surface this heat flux is balanced by the production of ice at a rate

$$\rho L \, dh/dt = Q_i \qquad (2)$$

where L is the heat of fusion, ρ is the density of ice, and t is time. Equating the two and integrating with the initial condition that $h = 0$ at $t = 0$ results in

$$h = \left(\frac{2k}{\rho L}\right)^{1/2} [(T_m - T_s)t]^{1/2} \qquad (3)$$

In (3) the bracketed term is the "square root of degree days of freezing" if the top surface temperature of the ice is taken as the air temperature. In practice, data show that an additional coefficient, α, usually in the range 0.5–0.8, must be applied to the right-hand side to give results as measured, and the measurements that have generally been used are usually for thicknesses greater than about 10 cm. There are other assumptions implicit in the above analysis, such as neglect of specific heat effects, but the major limitation is the assumption that the top surface temperature is the same as the air temperature.

PROPOSED METHOD

I first experienced the difficulty inherent in the result embodied in (3) while doing a numerical simulation of thin ice growth with small time steps. To do the simulation, (1) and (2) were used to determine the growth rate at zero time, which resulted in an infinite growth rate, a result that was clearly in error, since that would require an infinite heat loss. As will be shown below, inclusion of the effect of the thermal resistance between the top of the ice surface and the bulk temperature of the air removes this difficulty. More importantly, it provides an analytical result which is applicable for both thin and thick ice. The method entails only selection of a single coefficient which has been bracketed by the results from several data sets as well as energy budget analyses.

In addition to (1) and (2) above, the flux of heat Q_{ia} from the ice surface to the air above is expressed in the form of a bulk heat transfer coefficient H_{ia} applied to the difference between the top surface temperature of the ice and the air temperature above the ice, resulting in

$$-Q_{ia} = H_{ia}(T_s - T_a) \qquad (4)$$

If the flux of heat to the atmosphere is assumed equal to the flux of heat through the ice, then T_s may be eliminated using (1), (2), and (4). This results in
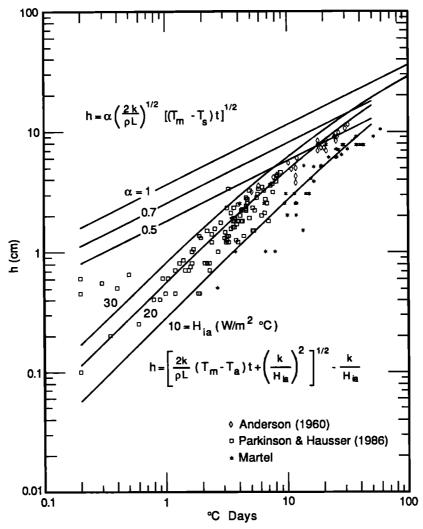
Fig. 1. Comparison of calculated thin ice growth with observations.

$$\frac{dh}{dt} = \left(\frac{1}{\rho L}\right) \frac{T_m - T_a}{\left(\frac{h}{k} + \frac{1}{H_{ia}}\right)} \qquad (5)$$

This may be integrated, again with the boundary condition that $h = 0$ when $t = 0$ and results in the expression for h in the form

$$h = \left[\frac{2k}{\rho L}(T_m - T_a)t + \left(\frac{k}{H_{ia}}\right)^2\right]^{1/2} - \frac{k}{H_{ia}} \qquad (6)$$

For large values of the product $(T_m - T_a)t$, (6) converges to the form of (3), and ice growth is proportional to $t^{1/2}$. For small thicknesses, however, ice growth is proportional to t but at a much lower rate than given by (2). To apply (6) in practical cases the bulk heat transfer coefficient must be estimated. One means of doing this would be to apply detailed energy budget methods to the top surface of the ice, calculate the net transfer Q_{ia}, determine T_s, and then determine H_{ia} by dividing by the temperature difference $T_s - T_a$. This requires many calculations and estimates of coefficients in the expressions for the energy budget components.

In practice, I have found a reasonable approximation is to use a constant value of H_{ia} with some consideration of wind speed. For still air conditions H_{ia} has been found to be in the

neighborhood of 10 W m^{-2} $°C^{-1}$; for conditions typical of say, the St. Lawrence River or the midwestern United States, a value of 20–25 has been found to be reasonable and for exposed windy conditions a value as high as 30 W m^{-2} $°C^{-1}$ has been found to be reasonable when the calculations are for several days duration.

COMPARISON OF THE TWO METHODS

Figure 1 shows calculated results for the two methods in a log-log format which has the advantage of showing clearly the power law dependence on t at various thicknesses as well as the convergence of the two methods at large times or thicknesses. For the traditional method, results for a coefficient α that must be applied to (3) of 1.0, 0.7, and 0.5 are shown in recognition of the experience that the coefficient is usually found to be about 0.6. The value of $\alpha = 1$ corresponds to assuming that the surface temperature of the ice is the same as the air temperature (or that the transfer coefficient is infinity for the proposed method). Values of $H_{ia} = 10$, 20, and 30 W m^{-2} $°C^{-1}$ are shown for the proposed method. It is clear that the two methods differ markedly for small values of $(T_m - T_a)t$ corresponding to small thicknesses, while at large values of $(T_m - T_a)t$ the two methods converge.

VALIDATION OF THE METHOD

Until recently I had few systematic data to provide a convincing presentation and contrast of the two methods. *Anderson* [1960] presented an analysis essentially equivalent to that described above although the thinnest ice of his data set was about 3 cm. Recently, two additional data sets have become available.

Parkinson and Hausser [1986] presented an extensive set of thin ice data obtained by following newly formed ice downstream on the St. Lawrence River. The water temperature was known to be at 0°C (thus the heat transfer to the underside was zero), and the thickness, air temperature, and time of travel to the location of measurement were determined. The data extend down to thicknesses a fraction of a centimeter. Finally, in experiments concerned with the freezing of sludge (which is mostly water) J. Martel (personal communication, 1988) obtained data on the thicknesses down to 1 cm. These three data sets are presented in Figure 1. Martel's data were obtained in a roofed enclosure but open on the sides and probably are representative of still air conditions, since the ice surface was somewhat protected from the wind. *Anderson*'s [1960] data were shown by *Adams et al.* [1960] to be bracketed by H_{ia} = 12 and 29 W m^{-2} °C^{-1}, corresponding to still air and a 6.7 m^{-1} wind speed, respectively. The extensive data of *Parkinson and Hausser* [1986] are for the most part bounded by H_{ia} = 10 and 30 except for a few data points at very small thicknesses and exposure times. These may be subject to measurement error or short-term energy budget effects not adequately accounted for by use of a simple heat transfer coefficient. For values of $(T_m - T_a)t$ above 1°C day almost all of their data lie in the 10–30 range of H_{ia}.

OTHER IMPLICATIONS

Aside from the obvious advantage of representing thin ice growth data well, the proposed method has certain other implications that are important in research on river ice processes. In some simulations in which the ice both grows and melts as the driving thermal inputs change, it is important to account for the thermal inertia represented by the formation of ice. The proposed method conserves energy, since it gives a growth rate consistent with the heat fluxes from the ice while the traditional "Stefan" formula results in growth rates that exceed the corresponding loss of heat. This

difference is important, particularly in situations where thin ice is alternately formed and melted. In other cases such as the formation of ice on slightly warm water, the actual ice growth is a result of the residual of the heat flux through the ice and the heat flux to the bottom surface. If the growth rate were to be calculated ignoring the thermal resistance associated with the air thermal resistance above, it is possible to predict ice formation when, in fact, the balance of heat fluxes is such that no ice forms.

NOTATION

h ice thickness, m.
H_{ia} heat transfer coefficient, W m^{-2} °C^{-1}.
k thermal conductivity, W m^{-1} °C^{-1}.
L heat of fusion, J/kg^{-1}.
Q_{ia} heat flux to atmosphere, W/m^{-2}.
t time, s.
T_a air temperature, °C.
T_m melting point temperature, °C.
T_s top surface temperature, °C.
ρ density of ice, kg/m^{-3}.

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