Resolution and uncertainty in spectral decomposition

Matt Hall*

Over the last five years or so, spectral decomposition has become a mainstream seismic interpretation process. Most often applied qualitatively, for seismic geomorphologic analysis (e.g., Marfurt and Kirlin, 2001), it is increasingly being applied quantitatively, to compute stratal thickness (as described by Partyka et al., 1999). It has also been applied to direct hydrocarbon detection (Castagna et al., 2003). Broadly speaking, we also apply the principles of spectral decomposition any time we extract a wavelet or look at a spectrum.

To date, most of this work has been done with the windowed (‘short-time’) Fourier transform, but other ways of computing spectra are joining our collective toolbox: the S transform (Stockwell et al., 1996), wavelet transforms, and matching pursuit to name a few (e.g., Castagna and Sun, 2006). In the literature, there is frequent mention of the ‘resolution’ of these various techniques. Perhaps surprisingly, Heisenberg’s uncertainty principle is sometimes cited as a basis for one technique having better resolution than another. In this article, we ask what exactly we mean by ‘resolution’ in this context, what the limits of resolution are, and what on earth quantum theory has to do with it.

A property of nature

Heisenberg’s uncertainty principle is a consequence of the classical Cauchy–Schwartz inequality and is one of the cornerstones of quantum theory. Here’s how Werner himself explained it:

‘At the instant of time when the position is determined, that is, at the instant when the photon is scattered by the electron, the electron undergoes a discontinuous change in momentum. This change is the greater the smaller the wavelength of the light employed, i.e. the more exact the determination of the position. At the instant at which the position of the electron is known, its momentum therefore can be known only up to magnitudes which correspond to that discontinuous change; thus, the more precisely the position is determined, the less precisely the momentum is known, and conversely.’

Heisenberg (1927), p 174-5.

The most important thing about the uncertainty principle is that, while it was originally expressed in terms of observation and measurement, it is not a consequence of any limitations of our measuring equipment or the mathematics we use to describe our results. The uncertainty principle does not limit what we can know, it describes the way things actually are: an electron does not possess arbitrarily precise position and momentum simultaneously. This troubling insight is the heart of the so-called Copenhagen Interpretation of quantum theory, which Einstein was so famously upset by (and wrong about).

Gabor (1946) was the first to realize that the uncertainty principle applies to information theory and signal processing. Thanks to wave-particle duality, signals turn out to be exactly analogous to quantum systems. As a result, the exact time and frequency of a signal can never be known simultaneously: a signal cannot plot as a point on the time-frequency plane. This uncertainty is a property of signals, not a limitation of mathematics.

Heisenberg’s uncertainty principle is usually written in terms of the standard deviation of position \( \sigma_x \), the standard deviation of momentum \( \sigma_p \), and Planck’s constant \( h \):

\[
\sigma_x \sigma_p \geq \frac{h}{4\pi} \approx 5.3 \times 10^{-34} \text{ m}^2 \text{kg s}^{-1}
\]

In other words, the product of the uncertainties of position and momentum is no smaller than some non-zero constant. For signals, we do not need Planck’s constant to scale the relationship to quantum dimensions, but the form is the same. If the standard deviations of the time and frequency estimates are \( \sigma_t \) and \( \sigma_f \) respectively, then we can write Gabor’s uncertainty principle thus:

\[
\sigma_t \sigma_f \geq \frac{1}{4\pi} \approx 0.08 \text{ cycles}
\]

So the product of the standard deviations of time, in milliseconds, and frequency, in Hertz, must be at least 80 (the units are ms. Hz, or millicycles). We will return to this uncertainty later.

If we can quantify uncertainty with such exactitude, what’s the problem? In part it is because there are at least two sides to what is usually called ‘resolution’. These are *sampling* and *localization*. These related concepts each apply to both the time and frequency domains.

Sampling

As scientists in the digital age, we are very familiar with the concept of sampling. It is analogous to resolution in digital photography, where it is usually measured in dots per inch (see Figure 1). It tells us how detailed our knowledge of something is.

---

*hallmt@conocophillips.com
In the same way that digital images are sampled in space, and digital signals are sampled in time, digital spectra are sampled in frequency. Intuitively, one might expect frequency sampling $\Delta f$ to depend on time sampling $\Delta t$. In fact, time sampling determines maximum bandwidth (the Nyquist frequency is half the sampling frequency), but it does not affect sample spacing in the frequency domain. For the Fourier transform, the only control on frequency sampling is the length of the analysis window (see Figure 2). Specifically, sample spacing in frequency $\Delta f$ is given by $1/T$, where $T$ is the window length in seconds. A 100 ms window means we get frequency information every $1/0.100 = 10$ Hz; reduce the window to 40 ms and the chunks are much bigger: $1/0.040 = 25$ Hz.

What is the practical consequence of, say, poor frequency sampling? For one thing, it can limit our ability to accurately estimate the frequency of a signal. The true frequency of a given signal is most likely between two samples (see Figure 3). The discrepancy is in the order of plus or minus one quarter of the sample spacing, or $1/4T$, and could therefore be large if the time window is short. For example, in Figure 3 the uncertainty is around ±4 Hz.

As well as hindering our estimation of the frequency of a single signal, sampling impacts our ability to determine whether two or more distinct frequencies are present in the signal. Following the analogy with the time domain, or with digital images, this type of discrimination is normally what we mean by 'resolution'. Figure 4 shows that, in order to be able to accurately resolve the frequencies of two signals, they must be about twice the sample spacing apart or more. Intuitively, we see that we need local maxima to be separated by a minimum. If they are closer than this, they may be ambiguous in the spectrum (the exact result depends on the window function and the actual frequencies). For the 64 ms Hann window shown in Figure 4, resulting in a 15.6 Hz sampling, we cannot use the Fourier transform to recognize the presence of two signals less than about 32 Hz apart. Any closer than this and they look like one broad spectral peak.

What about non-Fourier techniques? There is an increasingly wide acceptance of other spectral decomposition algorithms, especially the $\delta$ transform and various wavelet transforms (Castagna and Sun, 2006). Unfortunately, the jargon and mathematics behind these techniques sometimes get in the way. I strongly recommend Hubbard (1998) for those who are interested but perplexed; her book is very well written and accessible for any scientist.

![Figure 1 Resolution and localization in digital photography. (a) Sharp image at full resolution, 512 × 512 pixels, shown here at about 350 dpi. Very small-scale features are resolved. (b) The same image with a 25 × 25 pixel Gaussian smoothing filter applied. (c) The same image downsampled by a 10 × 10 pixel filter, to 51 × 51 pixels, or 35 dpi at this scale. (d) The Gaussian-filtered image in (b) after downsampling every 10th pixel. Sampling the smoothed image is not as detrimental as sampling the sharp image.](image1)

![Figure 2 Sampling and localization. (a) A 128 ms, 32 Hz signal, padded to 256 ms. (b) Its spectrum, computed by Fast Fourier Transform (FFT); the short window means that the spectrum is sparsely sampled, since $\Delta f = 1/0.256 = 3.9$ Hz. (c) The same 128 ms, 32 Hz signal, padded to 1024 ms. (d) Its spectrum; the longer window has improved the sampling in frequency ($\Delta f = 1/1.024 = 0.977$ Hz), but the uncertainty in the signal’s frequency is the same, indicated by the relatively wide central peak in the spectrum. (e) A 768 ms, 32 Hz signal, padded to 1024 ms. (f) Its spectrum, which has the same sampling as (d), but much better frequency localization. Adapted from Stockwell (1999).](image2)
Localization

Although arguably less tangible than sampling, the concept of localization is also familiar to geophysicists. Broadly speaking, it is analogous to focus in digital photography: a measure of how smeared out something is. Note that this is quite different from sampling resolution, since something can be smeared out, but highly sampled (as in Figures 1b and 2b).

Localization in the time domain is fairly intuitive: it depends on bandwidth, tuning effects, and how stable the data are in phase. It limits how accurately and consistently we can interpret a given reflectivity spike in the seismic.

Localization in the frequency domain depends on leakage. If we compute the Fourier transform of any finite signal, its spectrum displays the phenomenon known as ‘leakage’. For example, it is well known that the Fourier transform of a boxcar - an untapered, rectangular window - is a sinc function of infinite length (see Figure 6). The infinite sidelobes of the sinc function, and the fact that the peak is not a spike but is spread out a bit, constitute leakage. Windowing, the process of selecting a region of data to analyze, causes leakage; it is not an artifact of Fourier analysis per se.

So there are two distinct aspects to localization in the frequency domain: spectral sidelobes and peak width. Let’s look at each of these in turn.

Tapering the window reduces sidelobes. While the spectrum of the boxcar function, and of any signal windowed with a boxcar, has infinite sidelobes, they are much reduced for a tapered function (see Figure 6). This is known as apodization, literally ‘de-footing’. The most commonly used apodizing functions are the Gaussian, Hann, and Hamming functions, but there are many others. Like everything else in signal processing, there is a price to pay for suppressing the sidelobes: the spectral peak is spread out and reduced in magnitude. The degree of spreading and the reduction in magnitude depend on the specific function selected.

Figure 3 Measuring a single frequency. (a) Three 64 ms signals sampled at 1000 Hz with a frequency sample spacing of 1/0.064 = 15.6 Hz. The signals are 43 Hz (blue), 47 Hz (red), and 51 Hz (green) sine waves, with Hann windows applied. (b) The spectra are very similar, and would be unlikely to be discriminated, especially in the presence of noise.

One important aspect of these algorithms from the point of view of understanding resolution, is that they are extensions of Fourier analysis. The difference is in the analyzing function. In Fourier analysis and in the S transform, it is a Gaussian function; in wavelet analysis it is a family of wavelets at different scales. It is noteworthy that one of the original families of wavelets (the often-used Morlet wavelets) uses the Gaussian function as the wavelet envelope. It turns out that the difference with Fourier analysis is not really in the mathematics, but in the way this function - a window - is applied to the signal. You may have heard it said that wavelet transforms do not use a window; this is not strictly true.

Frequency domain sampling is less easily understood for these other transforms. What the S transform, wavelet transforms, and matching pursuit have in common is frequency-dependent resolution. That is, the sample spacing depends on the frequency. The easiest to grasp is perhaps the S transform (Stockwell et al., 1996). It is essentially a Fourier transform for which the window length varies with the frequency content of the signal, being short for high frequencies and long for low frequencies. This results in poor sampling for high frequencies, and good sampling for low frequencies (see Figure 5). Wavelet transforms take a more sophisticated (and computationally expensive) approach, but the principle is the same. These schemes are often collectively called ‘adaptive’ or, more descriptively, ‘multi-resolution’.

In a typical wavelet transform implementation, we end up with a sample at the Nyquist frequency $f_N$, another at $f_N/2$, then at $f_N/4, f_N/8$, and so on. (It is not quite as simple as this because wavelets do not have precise frequencies but ‘scales’ instead). Thus, for a seismic signal sampled at 500 Hz (2 ms sample rate), Nyquist is 250 Hz and the samples are at about 250, 125, 63, 32, 16, 8, 4, 2, and 1 Hz. The highest possible frequencies in this case would be around 0.8 × Nyquist = 200 Hz, and the lowest frequencies are usually close to 10 Hz. This means there would be perhaps five useful coefficients (incidentally, this explains why wavelet transforms are very useful for data compression: the signal can be faithfully reconstructed from relatively few numbers).

Figure 4 Resolving two signals. (a) A 64 ms signal sampled at 1000 Hz with a frequency sample spacing of 1/0.064 = 15.6 Hz. The signal consists of a 32 Hz and a 63 Hz sine wave, with a Hann window applied. (b) The spectrum does not resolve the two frequencies. (c) A 32 Hz with a 65 Hz signal, also with a Hann window. (d) The signals are more than twice the frequency sample spacing apart and are resolved.
Spectral peak width is the other key component of localization. It is affected by the taper function but primarily controlled by the signal itself. It is approximately equal to \(2/S\), where \(S\) is the length of the non-zero part - the ‘support’ - of the signal, not the length of the window. In the case of seismic reflection data, we are dealing with an overlapping train of signals representing reflections from subsurface strata. The length \(S\) of each individual signal - the ones we are trying to characterize - is equal to the length of the seismic wavelet. How long is that? In theory it is infinite, but in practice the energy falls below the noise level at some point. Experience suggests that wavelets are effectively in the order of 64–128 ms long. So for a typical seismic dataset, the width of the spectral peak is about 16–31 Hz, with the larger uncertainty for the shorter wavelet. Of course, if the windowing function is shorter than the seismic wavelet, then the peak is even wider because a maximum cannot be less than two samples wide. We see that there is no benefit to making the window shorter than the seismic wavelet.

It is instructive to examine the effect of increasing window length on the spectrum of a short-lived signal, and comparing to the same window operating on a longer-lived signal (Figure 2). We find that signals with short support are poorly localized in frequency. This frequency smearing has nothing to do with windowing: it is a property of nature.

**Measuring uncertainty**

The relationship between time and frequency resolution is illustrated in Figure 5 with ‘Heisenberg boxes’. Gabor called these boxes ‘logons’, which is preferable since we are dealing with Gabor’s uncertainty principle. These idealized cartoons show how the time-frequency plane can be tiled with rectangles of size \(\sigma_t \times \sigma_f\), where \(\sigma_t\) is the standard deviation of the time estimate and \(\sigma_f\) is the standard deviation of the frequency estimate. As we saw earlier, this product must be at least \(1/4\pi\). In practice, this lower bound is usually not reached by time-frequency transforms, though some get closer to it than others. It would be interesting to see researchers publish the areas of the Gabor logons with which their algorithm tiles the time-frequency plane.

It is instructive to consider a specific example of Gabor uncertainty. If you know the time localization to within \(\pm 5\) ms, say, then the frequency localization is the reciprocal of \(\pm 0.005 \times 4\pi = \pm 16\) Hz. It is not possible to know the frequency...
of a signal more accurately than this without compromising the timing accuracy. Uncertainties of this magnitude are important to us as seismic interpreters, especially if we are using quantitative spectral methods.

The effect of sampling is also measurable. Imagine you are trying to use spectral decomposition for a quantitative analysis of thin beds in your seismic. As discussed above, the length of the window you select determines the sample spacing in the frequency domain. If you select a 50 ms window, for example, then you will have information about the 20 Hz response, the 40 Hz response, the 60 Hz response, and so on. Any other information (that is, other slices in the so-called ‘tuning cube’), is unknowable with this window length and is therefore essentially interpolated data. A 20 ms window, which offers improved temporal resolution, provides you with spectral information at 50 Hz, 100 Hz, 150 Hz, etc. In striving to probe those thin beds, you may have ended up with only three or four data points within the bandwidth of your data. Any thickness estimates cannot hope to do better than ‘thin’, ‘thick’, and ‘thicker’. Chances are, you can interpret the ‘thick’ and ‘thicker’ beds with conventional interpretation of reflectors.

Conclusions

The spectral analysis of signals is governed by Gabor’s uncertainty principle, itself derived from the Cauchy-Schwartz inequality and analogous to Heisenberg’s uncertainty principle. Rather than limiting what we can measure, it describes a fundamental aspect of reality: signals do not have arbitrarily precise time and frequency localization. It doesn’t matter how you compute a spectrum, if you want time information, you must pay for it with frequency information. Specifically, the product of time uncertainty and frequency uncertainty must be at least $1/4\pi$.

For good localization in time, use a short window $T$ for Fourier methods (for wavelet transforms, analyze with small-scale - high frequency - wavelets). When you do this, you increase the frequency sampling interval, so you lose frequency resolution. Because of this sampling, you cannot tell exactly what frequency a signal is ($\pm 1/2T$), and you cannot discriminate similar frequencies (closer than $2/T$).

For good localization in frequency, use a long window (or analyze with large-scale wavelets). When you do this, you decrease the frequency sampling interval and get better frequency resolution, but the long window results in poor time localization.

Regardless of Fourier windows or wavelet scales, short-lived signals have wide spectral peaks and thus poor frequency localization. Only long-lived signals have narrow spectral peaks and correspondingly good frequency localization. This is the essence of Gabor’s uncertainty principle.

Acknowledgements

Grateful thanks are due to Tooney Fink, Sami Elkurdi, and Peter Rowbotham for their insight, and to an anonymous reviewer for several important improvements.

References


