We define spatial structuring as the mental operation of constructing an organization or form for an object or set of objects. It is an essential mental process underlying students’ quantitative dealings with spatial situations. In this article, we examine in detail students’ structuring and enumeration of 2-dimensional (2D) rectangular arrays of squares. Our research indicates that many students do not “see” the row-by-column structure we assume in such arrays. We describe the various levels of sophistication in students’ structuring of these arrays and elaborate the nature of the mental process of structuring.

Array. An orderly, often imposing arrangement…. A rectangular arrangement of quantities in rows and columns, as in a matrix. (American Heritage Dictionary, 1992)

The cognitive organism shapes and coordinates its experience, and in doing so, transforms it into a structured world. (von Glasersfeld, 1995, p. 57)

One of the fundamental goals for mathematics education researchers is to understand the nature, development, and range of mathematical thinking used by students. To achieve such understanding, we must not only describe students’ mathematical constructions and ways of reasoning, we must attempt to determine what mental processes make them possible. One mental process we have found to be critical in understanding students’ dealings with quantitative spatial situations is structuring (Battista & Clements, 1996). In this article, we examine in detail students’ structuring and enumeration of 2D rectangular arrays of squares.

SPATIAL STRUCTURING

We define spatial structuring as the mental operation of constructing an organization or form for an object or set of objects. Spatially structuring an object

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determines its nature or shape by identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. Structuring is a form of abstraction, which we take to be the process by which the mind selects, coordinates, unifies, and registers in working memory a set of mental items or actions that appear in the attentional field. There are different levels of abstraction of an item or action, including (a) isolating it in the experiential flow and grasping it as a unit, (b) internalizing it, which registers it in working memory so that it can be re-presented or “replayed” in its absence, and (c) interiorizing it, which purges an internalized item of its original sensory content so that it can be utilized in a novel situation (Steffe & Cobb, 1988; von Glasersfeld, 1995). As a form of reflective abstraction, structuring takes previously abstracted or structured items as content and integrates them to form new structures, which themselves can be taken as content in further acts of abstraction or structuring (von Glasersfeld, 1995). Structuring combines into stable patterns, not specific sensory input, but the mental actions that an individual uses to link particles of sensory experience. Indeed, Piaget and Inhelder (1967) claimed that the abstraction of shape “rests upon an active process of putting [things] in relation, and it therefore implies that the abstraction is based on the child’s own actions and comes about through their gradual co-ordination” (p. 78).

The essential role that structuring can play in quantitative spatial situations is illustrated by 2 third graders who were trying to count the edges of a triangular prism they had built from rods joined at their ends (Battista & Clements, 1996). The students each counted the rods one at a time, but in an unorganized fashion, getting a variety of answers they could not agree on. Only after an interviewer asked one of the students to count slowly to “show us what you are doing” did the student correctly enumerate the rods, deliberately counting the three rods on each base, then the three lateral rods. These students were unable to correctly enumerate the edges until they spatially structured the elements of the configuration into three spatial composites of 3—the two triangular bases and the set of three lateral rods.

In a previous study, we investigated the spatial structuring students employed while they enumerated 3D arrays of cubes configured in rectangular prisms (Battista & Clements, 1996). We found that spatial structuring preceded meaningful enumeration. That is, students’ spatial structuring provided the input and organization for the numerical processes they used to enumerate an array of cubes. In particular, students who spatially structured an array into columns or layers organized their enumeration by the number of cubes in a column or layer, often skip-counting or multiplying to find the total. Alternatively, students who structured an array as an uncoordinated set of prism faces determined the number of cubes by enumerating cube faces on all or some of the prism faces, often counting cubes along the prism’s edges more than once.

Using several theoretical frameworks (e.g., Cooper, 1990; Morss, 1987), we described the underlying mental processes and cognitive milestones in the
development of students’ spatial structuring of 3D rectangular arrays of cubes. Initially in this development students see arrays as unorganized sets of cubes. When they attempt to organize the cubes, their structuring is local rather than global. That is, students construct spatial composites that consist of small sets of cubes—a prism face or portion thereof, a row or column of cubes. But they have no global scheme for organizing these composites. At this stage, students’ conception of 3D arrays is often an uncoordinated set of views of orthogonal prism faces. Students progress beyond this stage when they construct the notion of perspective, recognize that the orthogonal views must somehow be coordinated, and become capable of accomplishing such coordination. The coordination process enables students to properly relate orthogonal prism faces so that cube faces depicting the same cubes can be recognized as such. To construct a proper global structuring of the array, however, students must integrate the orthogonal views into one coherent mental model of the entire array, inside and out. Such a global structuring consists of a scheme for iterating rows, columns, or layers of cubes, with layer structuring being the most efficient, both computationally and mentally.

When we continued to examine the relationship between students’ spatial structuring and enumeration of objects in 3D rectangular arrays, we found them having structuring difficulties even within single layers of cubes. The nature of these difficulties suggested that the relationship between students’ structuring and enumeration of two-dimensional rectangular arrays be investigated. Such structuring is essential for the development of the notion of area and is intimately related to multiplication because 2D arrays are a major model for and application of multiplicative thinking. In fact, Simon and Blume (1994) have reported preservice teachers having structure-related enumeration difficulties with a problem in which a nonsquare unit was used to measure area. Outhred and Mitchelmore (1992) found that as the row-by-column structuring evidenced in elementary students’ drawings of rectangular arrays of squares increased, so too did their use of multiplicative strategies in enumerating the squares. And Vergnaud (1983) found seventh graders having difficulties relating their use of multiplication to a proper spatial structuring of rectangular arrays of squares.

The goal of the present investigation is to extend our analysis of spatial structuring by examining in detail students’ structuring of 2D rectangular arrays of squares. By investigating students’ structuring of 2D arrays—a situation that is similar to, but simpler than, 3D arrays—we hope to better understand, and be able to elaborate on, the structuring process.

PROCEDURE

As the first step in our investigation of students’ development of 2D structuring, we interviewed students so that we could make distinctions in their ways and means of operating (Steffe, 1996). Analysis of these interviews resulted in a description of the various levels of sophistication in students’ structuring of 2D
arrays as they progress on the long and difficult road to constructing the row-by-column structuring used by sophisticated members of our culture. Our second step was to develop descriptions of the mental mechanisms students used in structuring, focusing especially on transitions between levels.

Because we were interested in the early development of students’ structuring of rectangular arrays of squares, we interviewed primary-grade students—12 second graders (ages 7–8 years)—during the fall. On three separate occasions we selected and individually interviewed 4 second graders for two or three 45-minute sessions each, using a subset of the tasks shown in Figures 1a–m. Different sets of tasks and administration orders were used for the second and third rounds of interviews so that we could pursue conjectures formulated from preliminary analyses of earlier rounds. To further investigate ideas and questions generated from the fall interviews, near the end of the school year we conducted a fourth round of individual interviews, this time giving all 12 of the original students the rectangle tasks shown in Figures 1n–q. Neither the research team nor the students’ classroom teacher instructed students on enumerating rectangular arrays during the school year. We also describe two particularly relevant episodes from separate interviews with a third grader and a fourth grader whose array work came to our attention.

For most problems, after being shown how a plastic square inch tile fit exactly on one of the graphically indicated squares in the rectangle, students were first asked to predict how many squares it would take to completely cover the inside of the rectangle (the \textit{original prediction}). Next, students were asked to draw where they thought the squares would be located on the rectangle, then to predict again how many squares were needed (the \textit{drawing prediction}). Finally, students covered the rectangle with square tiles and determined again the number of squares needed. The interviews were videotaped, transcribed, and analyzed in an attempt to make sense of students’ structuring of the arrays.

\section*{RESULTS}

\subsection*{Levels of Sophistication in Students’ Row-by-Column Structuring}

We now describe the different levels of sophistication that we ascribed to students’ structuring of rectangular arrays. For the most part, these levels apply to students’ structuring during original or drawing predictions, though sometimes, to corroborate our discussion of predictions, we describe the structuring students used to enumerate tiles. Unless noted otherwise, the students described are second graders.

Note that the level of sophistication exhibited by a student on a particular problem should not be construed as “the level of sophistication of the student” in some general developmental scheme. Instead, the levels describe students’ functioning on rectangular array tasks. Furthermore, because a student might exhibit slightly different levels of sophistication on different problems, several
Students were shown that 4 squares go across the top and 3 down the left side. Students were shown that 4 squares go across the top and 4 down the left side.

Students were shown that 5 square tiles go across the top (then the tiles were removed), and that 7 square tiles go down the middle (then the tiles were removed).

**Figure 1.** Rectangle interview tasks: How many squares does it take to completely cover the inside of the rectangle? (All dimensions are inches.)

Problems must be given to get some feeling for a student’s current level of functioning on these tasks.
Level 1: Complete Lack of Row- or Column-Structuring

The student makes no use of a row or column of squares as a composite unit. He or she has difficulty both visualizing the location of squares in an array and counting square tiles that cover the interior of a rectangle.

Example 1. For Figure 1a, CS pointed and counted as shown in Figure 2a, predicting 30. When checking her answer with square tiles, she first counted the tiles as she pointed to them with her finger as shown in Figure 2b, getting 30. But she got confused, so she counted the tiles again, getting 24 once, then 27. CS was not able to count correctly until the interviewer asked her to count the tiles by placing them one-by-one into a bag. On the next task (Figure 1b), CS counted as shown in Figure 2c for her original prediction. For the task shown in Figure 1n, CS made her original prediction by counting as shown in Figure 2d, using 2 fingers together to count squares 1–6 and 22–25, arriving at a total of 30. After drawing the squares, CS counted them aloud, getting confused when, as she was about to count 32, she saw that she had exceeded her prediction of 30 (Figure 2e). Notice that CS drew nonequivalent rows and double-counted each square in rows 2–4 of the right column. When CS numbered the squares with a pencil while she recounted them, she arrived

\[ \text{Figure 2. Work of CS.} \]
at a total of 30, counting each drawn square once and only once (Figure 2f). Finally, after CS correctly covered the rectangle with square tiles, she again miscounted them. It was not until the interviewer separated squares into columns that CS was able to count correctly.

When attempting to enumerate squares that cover rectangular regions, CS was unable to adequately structure and count the squares when she tried to visualize them, draw them, and even when she physically placed them in the correct positions. She had great difficulty visualizing and counting squares that were not wholly or partially drawn. She was unable to keep track of what she had already counted.

However, although inaccurate, CS’s counting of undrawn squares was not random; there was some organization to it. For instance (see Figure 2c), she first counted the predrawn squares, then counted 9, 10, 11, 12, 13 down the right side and an equivalent number (15, 16, 17, 18, 19) up the left side of the interior. Several times she counted 3 squares horizontally: 13, 14, 15; 19, 20, 21; 23, 24, 25. Furthermore, CS’s overall counting of the squares formed a clockwise spiral. Thus, it is not that CS completely failed to structure the squares. Instead, her structuring was inadequate for the task of enumeration.

Example 2. In drawing squares to cover Figure 1c, BH drew the perimeter squares first, sometimes using the hash marks as guides, sometimes using his fingers to measure the height or width of a square (Figure 3a). He then drew the 6 squares in the interior. When asked how many squares fit in the rectangle, BH counted the squares in the perimeter of his drawing, getting 13. He then continued counting the interior 6 squares by ones, getting a total of 19. BH then checked his answer with square tiles. As he had in the past, he placed the tiles around the perimeter in a clockwise fashion first, then filled the interior. He counted the tiles as shown in Figure 3b (double-counting the last tile). BH then recounted the tiles, again double-counting a tile, even though he counted in a different order (Figure 3c).

Although BH’s drawing seemed to have some organization, his structuring was local, not global. He visually estimated the location of and drew each individual square, utilizing the hash marks to help him draw. But there was a lack of global coordination in BH’s activity. Indeed, despite his apparent repetition of
parts of the array (the second and third interior rows of 3), BH did not compare the number of squares in full rows; he ignored the inconsistency presented by the 3 squares over 4 in the middle of the bottom portion of his drawing. Furthermore, regardless of the fact that BH sometimes explicitly tried to connect the side of the square he was drawing to appropriate intersections of other sides, he did not seem disturbed when these segments did not properly connect. Even when BH counted tiles that he had correctly placed on the rectangle, his lack of appropriate structuring caused him to double-count some tiles. Both in his drawing and in his counting, BH seemed to structure the rectangular array in terms of its perimeter and interior.

Example 3. When AT was asked to predict the number of squares that cover the rectangle in Figure 1b, he first predicted 18 by visualizing, but not pointing to, squares. When asked to show the squares he was counting, AT pointed and counted as shown in Figure 4a, getting 16. In drawing the array, AT made the 3 squares in the second row by moving down and to the left for each. He completed the missing part of the array by correctly drawing two vertical and two horizontal segments. AT enumerated the squares in the drawn array by numbering each square (Figure 4b).

![Figure 4. Work of AT.](image)

Although AT’s weak structuring caused him difficulty in visualizing the location of squares in his original prediction, his drawing suggests that he was able to utilize the graphic cues to correctly structure his drawing activity. And once he had drawn the array, he had no difficulty enumerating the squares in it because he wrote the numbers in each square, eliminating the chance of double-counting and missing squares, and, indeed, the need to keep track of his counting path.

**Level 2: Partial Row- or Column-Structuring**

The student makes some use of a row or column as a composite unit, but this composite is not used to cover the whole rectangle.
In this task (which was given on the same day but not immediately after the task in Example 2) BH started to restructure the arrays. Although he did not use a viable (if any) structure for the interior of the rectangle, the fact that after counting 6 squares in the bottom row he was able to infer that the top row had 6 suggests that he was able to treat the bottom as a composite unit. After correctly drawing squares in the rectangle by deliberately joining segments to hash marks, BH correctly enumerated the squares (again as top, bottom, sides, and middle) providing further evidence that he was structuring the array into components.

**Level 3A: Structuring an Array as a Set of Row- or Column-Composites**

The student conceptualizes the rectangular array as being completely covered by copies of row- or column-composites but does not properly coordinate those composites with the orthogonal dimension.

Example 4. (See Figure 1g.)

**BH:** First I count the bottom and there’s 6. So the top and bottom would equal 12 [moving his hands inward as shown in Figure 5]. And these 2 [on the right and left sides, in the middle] would be 14. I’d say maybe 12 in the middle [using fingers to visually estimate the placement of squares one by one]; 12 plus 12 equals 24. So I’d say 24.

In this task (which was given on the same day but not immediately after the task in Example 2) BH started to restructure the arrays. Although he did not use a viable (if any) structure for the interior of the rectangle, the fact that after counting 6 squares in the bottom row he was able to infer that the top row had 6 suggests that he was able to treat the bottom as a composite unit. After correctly drawing squares in the rectangle by deliberately joining segments to hash marks, BH correctly enumerated the squares (again as top, bottom, sides, and middle) providing further evidence that he was structuring the array into components.

**Level 3A: Structuring an Array as a Set of Row- or Column-Composites**

The student conceptualizes the rectangular array as being completely covered by copies of row- or column-composites but does not properly coordinate those composites with the orthogonal dimension.

Example 5. (See Figure 1i.) (Int. is the interviewer.)

**MO:** [Nodding while she goes across rows, counting by ones] I guess 20.

**Int:** How did you get it?

**MO:** Because I counted each row in 4s. I didn’t count them in 4s, but 4 in each row.

**Int:** Do you want to draw where you saw them?

**MO:** [She draws the squares individually from left to right, beginning with the top row; see Figure 6a.] I counted 20, but when I was drawing the 4 in the row, … I didn’t get 20…. I got 16.

**Int:** How did you see them before, when you got 20?

**MO:** I think I saw them and *I thought that they were smaller.*

**Int:** [After MO finds that 12 square tiles cover the rectangle] Why do you think that was different than the 16 you drew?

**MO:** I’m not really sure; because maybe I drew them smaller or bigger…. I did draw them smaller.
MO attempted to fill the rectangle with rows of 4 squares. However, she ignored the given column dimension (3 squares down), drawing each row-composite by visually estimating how far it extended downward. She used a similar visual estimation in the following example.

Example 6. (See Figure 6b.)

MO: Here there would be 1, here 2, 3, here 4 [continuing to point as indicated in Figure 6b] 32.

Int: How did you know where to count?

MO: Because I just like tried to think of what size the square was.

The inaccuracy of MO’s estimation of the vertical placement of squares, coupled with the accuracy of her horizontal placement, suggests that she intentionally maintained a row-of-4-composite but ignored the markings that indicated the vertical placement or number of rows. The imagery that MO used was not properly constrained by relevant information given in the problem because her mental model of composing the whole array out of a row-composite of 4 did not include distributing this composite over the squares in a column.

Level 3B: Visual Row- or Column-Iteration

The student iterates a row-as-composite by distributing it over the elements of a column. When either drawn squares or square tiles are available, the student uses them to index the iteration. When perceptual material is absent, the student determines iterations by visually estimating how the rows fit in the rectangle.

Example 7. (See Figure 6q.)

JH: Five, 10, 15, …, 45 [motioning across rows inside the rectangle].

Int: How did you get that?

JH: I was trying to guess where the bottoms of the square were. [JH then places 7 squares down the right column and quickly points to each and immediately says 35] 5, 10, 15, …, 35. Five times 7. I’m positive.

Int: How are you positive?

JH: Five in each row [pointing at the rectangle].
Initially JH organized the squares into rows of 5 and used visual estimation to iterate this row-as-composite. But he recognized that he was just “guessing,” that he could not keep track of his row iteration without placing squares in at least one column. His enumeration scheme consisted of iterating rows by distributing them over the elements in a column, but he could not yet implement it without appropriate perceptual material to indicate column squares. In the next two tasks, AT, like JH, was able to iterate composites of 5 when perceptual material was available.

Example 8.

AT: (See Figure 1p.) I went 1, 2, 3, 4, 5 [moving his finger across row 3 while he counts; see Figure 7a]. So, I can count by 5s; 5, 10, 15, 20 [pointing to visible squares; see Figure 7b].

AT: (See Figure 1o.) Well, I’d probably count by 4s; [sweeping his finger across the rows while he counts] 4, 8 (long pause), 12 (pause), 16 (pause), 20 (pause), 24. (See Figure 7c.)

Int: How did you get those numbers? Were you doing anything with your fingers?

AT: Yeah. After I got to 8, I started using my fingers.

Int: Show me with your fingers what you did.

AT: After 8 I’d go, [using his fingers to show how he counted on them] 9, 10, 11, 12. [AT then illustrates how he used his fingers to find subsequent multiples of 4.]

In both tasks, AT was able to properly structure the rectangular array into rows by indexing his iteration of rows with an appropriate set of squares. In the second task, he knew that there were rows of 4 but needed his fingers to implement his skip counting. (As illustrated by the second task in Example 8, in our classification scheme we do not require that a student have memorized...
all the numbers in a skip-count sequence to say that he or she is iterating a composite unit.)

As the next examples illustrate, however, even though AT could use row-by-column structuring when visual material was present, he was unable to perform such structuring without sufficient perceptual material.

Example 9. (See Figure 1j.)

AT: [AT moves the end of a pencil across the rectangle three times as indicated in Figure 7d.] If this would be 4 [A], and then there’d be another 4 [B], down here there’d be another 4 [C]. Maybe four 4s.

Int: Why do you say “maybe four 4s”?

AT: [AT moves the end of his pencil across the rectangle again.] Because it could be three. [He starts over.] One [row of 4] right here [A], 2 right here [B], and 3 right here [C]. But, there might be another one down there [moving his pencil back and forth several times at location D.]

It is clear from AT’s language that he took a row of 4 squares as a spatial composite. But he iterated this composite visually, ignoring the fact that the interviewer had demonstrated that there were 4 squares from top to bottom in the rectangle.

Example 10. (See Figure 1q.)

AT: Since 5 go across here [sweeping his finger horizontally across the rectangle 2 times], I could count by 5s, but they’d have to be equal up with one of these [pointing to a tile on the desk].

Int: How can you count by 5s then?

AT: I could remember how big those are [pointing to a tile] and then go across with my finger. [AT traces a segment with his finger across the upper portion of the rectangle.] Just like a pencil; [placing a pencil horizontally and using it to move down the rectangle slowly while he counts] 5, 10, 15, 20, 25, 30, 35.

Int: So, you think this tile is about that size. [The interviewer places a tile in the upper left corner, above the pencil, as in Figure 7e. AT nods his head yes.] How did you know to move it down as many times as you did? Do you know how many times you went down?

AT: About um, 7.

Int: Were you trying to go down that many times or were you just kinda judging?

AT: Yeah, judging.

Int: Go ahead and draw and then we’ll make another prediction.

[AT draws four vertical lines from top to bottom then five horizontal lines left to right (see Figure 7f). He places his pencil tip on the left side of the rectangle midway between the last line he drew and the bottom of the rectangle. He is about to draw another horizontal line, but stops.]

AT: That wouldn’t fit; I can see that it would be too small.

Int: [Motioning down the third column] How many did I put down the middle [referring to the introduction of the task]?

AT: Seven. [AT points while he counts squares down the third column in his drawing] 1, 2, 3, 4, 5, 6. I guess I skipped one line. [He adds a horizontal segment to his
drawing to make seven rows, then places tiles on the rectangle and counts one-by-one to 35.]

Int: Is there another way you could count without counting by ones?

AT: By 7s and um, 5s.

Int: Which way would you have to count by 5s?

AT: [Moving his fingers across rows 2 and 3] Across.

Int: Can you do it by 5s?

AT: Five, 10 [moving a finger over a portion of rows 1 and 2 for each number], 15 (slight pause), 20, 25, 30, 35 [while he counts, he points to successive squares down column 2].

Int: And this, when you point like this [indicating the second tile in row 1], means what?

AT: It means the whole row of 5s.

Clearly AT was trying to iterate a composite of 5. But when there was no perceptual material available, he used visual estimation to determine how many iterations there would be (as is verified by his drawing activity). He, like MO, did not incorporate the number of squares in a column into his visual iteration process. Initially, AT was not even sure how many times he had iterated the row of 5, which he would certainly have been aware of had he explicitly used the number of squares in a column to determine his iterations of rows. Even his drawing was done visually. However, when the interviewer asked, “How many did I put down?” AT seemed to recognize, perhaps for the first time, the connection between the number of squares in a column and the number of rows.

So AT seemed to have a clear row-structuring for the array. But a recognition of how the given information about squares in a column could be used to guide his row iteration was just emerging. At this time, he could not “step out of and look down on” the iteration process so that he could analyze it and properly relate it to the given column information. AT had not yet interiorized the process of distributing a row over the elements of a column.

Combining rows or columns into composite units. Some students combined rows or columns, then iterated the resulting composite of composites. For example, when enumerating squares that he had correctly drawn in a 6-by-5 rectangle, PT counted the 5 squares in the first column, said 10 for the first 2 columns, then iterated this two-column composite of 10, saying 20, 30, to find the total. PT’s formation of a “composite of composites” required a higher level of abstraction than that required to merely structure a rectangular array in terms of rows or columns. He seemed to be able to “look down” on his initial row structuring of an array, analyze it numerically, then restructure it. He was able to spatially restructure the array to make enumeration more efficient. Sometimes, however, this approach can lead to difficulties. In fact, several times, JD incorrectly enumerated the composites of composites he formed (see Example 22).

Level 3C: Row-by-Column Structuring; Iterative Process Interiorized

The student iterates a row or column, using the number of squares in an orthogonal column or row to determine the iterations. The original perceptual
material (e.g., drawn squares or square tiles) is not used during iteration.

Example 11. (See Figure 1o.)

PT:  I’m thinking that I could count by 4s right now.
Int:  Where do you see that?
PT:  [Drawing a row of 4 squares] There are 4 boxes—1, 2, 3, 4—on top.
Int:  So, how many do you predict will be in the whole rectangle?
PT:  I could use 4 times (pause), but I have to know how many there are. [Counting and tapping on the squares in the right-hand column] 1, 2, 3, 4, 5, 6. Four times 6 equals (pause). [PT now uses both hands to count. On one hand he counts the number of 4s; on the other hand he counts by ones up to 4.] I put one finger up for counting it as [one] 4, and then I counted to make sure I was using 4 with my fingers with my other hand; 29.

Although PT made a counting error, the level of sophistication of his spatial structuring is quite high. He saw immediately that he could compose the whole array as rows of 4 squares. Furthermore, he did not need to look at the array to enumerate it. He simply needed to know the number of rows, which he determined by counting the squares in a column.

At this point PT had abstracted a spatial structuring of the array as “6 rows of 4.” This abstraction was at a high enough level that he could leave behind the perceptual material in the rectangle and could proceed with the task of evaluating the numeric problem of finding six 4s. PT had interiorized the spatial structure of the array, verbally describing it as “4 times 6.” He did not need to re-present either the spatial configuration of 6 rows of 4 or the process of iterating the rows to hold onto this structure. He needed his fingers only to implement his enumeration of six 4s.

Example 12. Subsequently, PT was given the task shown in Figure 1q.

Int:  [Placing square tiles down the middle column] 7 squares fit down the middle, [after removing the 7 tiles] and 5 squares fit in a row across the middle [placing tiles]. How many squares do you think it will take to cover the whole rectangle?
PT:  You said 7 up, right? Five across. [Counting and pointing across the top row by ones] 5 across, 7 down; (pause) 7 down, 5 across. [Motioning from left to right across the top 3 rows of the rectangle] 5, 10, 15.
PT:  [Counting multiples of 5 on his fingers] 5, 10, 15, 20, 25, 30, 35; 35.
Int:  How did you know to stop at 35?
PT:  Because when I lifted up my last finger I knew that was 7.
Int:  What did that tell you? Why 7?
PT:  There are only 7 down that way [motioning vertically down the middle of the rectangle].

This problem seemed to temporarily stump PT. Three times he repeated “5 across and 7 down,” as if he were trying to figure out exactly how to use this information that he knew was essential. Finally, he seemed to make an important discovery: He saw that iterating the rows required him to count by 5s and
that the number of rows was indicated by the number of squares “down.” Implementing his newly created enumeration scheme, however, required him to generate some of the missing perceptual material, thus the sweeps for the top three rows. These sweeping actions enabled him not only to interiorize the creation of rows but to take the elements of the composite of 7 as symbols for successive row iterations; he no longer had to actually re-present the rows. Because PT had interiorized the 7-rows-of-5 structure, he could leave the original perceptual material behind and focus on the task of numerically evaluating seven 5s. He used his fingers not to re-present 7 rows of 5 squares, but to keep track of his counting by 5s.

Example 13. JH is asked how many square tiles it would take to completely cover an 8-by-11 rectangle. After he places and counts 8 tiles along the top of the rectangle, he removes them, then places and counts 11 tiles down the right side. He smiles and says “8 times 11,” then adds 11, eight times.

In this episode, we see that JH not only used row-by-column structuring without the aid of perceptual material, he actually superimposed this structure on a rectangle without external guidance. The latter action is a good indication that he, like PT, had interiorized the row-by-column structure. But JH demonstrated a higher level of abstraction than PT because JH applied his structuring in the absence of explicit statements about how squares would fit in the rectangle.

Constructing Equivalent Rows

An essential component of constructing a row-by-column structure for 2D arrays is understanding that the array’s rows (or columns) are necessarily equivalent. To assess students’ understanding of row equivalence specifically, we asked students first to cover the top row of a 7-by-3 rectangle with square tiles, establishing that it took 7 squares to do this. We then asked students to predict how many square tiles it would take to cover the second row, then, after they checked their answer with tiles, to predict for the third row.

Some students inferred that the rows would necessarily be equivalent: “Because they’re the same size and same length.” Other students said that there were 7 squares in each row but did not seem to see it as a logical necessity. Instead, they visually estimated how many squares fit in a row. CS’s response illustrates that such visual estimates could be mistaken.

Example 14.

CS:  [Using a finger to imagine and count the second row of squares as shown in Figure 8a] 7.
Int:  All right. I want you to use the squares and check your answer for the second row.
[CS checks by sliding the square tiles in the first row down to cover the second row, then returning them to the first row.]
Int: How many squares do you think it will take to cover this row [pointing to row 3]?
CS: [Using her finger to count as shown in Figure 8b] 8.

![Figure 8](image.png)

Figure 8. Work of CS.

CS did not see that the rows were necessarily equivalent, even though she moved the set of squares in row 1 into row 2, an action that seemed to help some students construct the row structure of the array. However, it seemed that CS moved this set of squares down not because she saw that they should fit spatially but because she recalled that it had the same number of squares as she had predicted for row 2.

Interestingly, students who made a clear logical inference on this task tended to structure subsequent arrays into equivalent rows or columns. Students who did not see the equivalence on this problem, or who used visual estimation, often structured subsequent arrays as having nonequivalent rows or columns.

Transitions Between Levels

One-Dimensional Paths as a Starting Point

According to Cobb and Steffe (1983), to count objects requires that one coordinate the production of a sequence of number words with the production of a sequence of unit items. But to put the unit items into a sequence requires one to organize or structure them. Initially this structuring is one-dimensional — while a child counts, she follows a path defined by her sequence of counting acts. For the count to be correct, however, the child must somehow monitor the path that is being followed. She must be sure that the path touches each object once and only once. The difficulty for CS and other students exhibiting her level of sophistication on the 2D-array enumeration tasks was that they could not adequately monitor the paths they followed during counting. We conjecture that the reason such students could not keep track of their paths was because their structuring was one-dimensional instead of two-dimensional. That is, to keep track of a counting path for objects located in 2D space requires children to progress beyond the act of following the path; they must adequately situate the path in 2D space.

Moving From 1D to 2D Local Structuring

The following two examples illustrate how students progress beyond a strictly
1D path-structuring to a structuring of composites of squares situated within 2D space. These examples also demonstrate how structuring is based on action.

Example 15. Fourth grader DA was asked to cover a 4-inch-by-4-inch square using square-inch tiles (P. Wilms, personal communication, December 1, 1993). DA placed the tiles clockwise around the perimeter, then filled in the middle (as indicated in Figure 9a). Several days later, DA was shown a 4-by-4 grid and was asked to find its area. He said it was 16: “I counted and added 4 plus 3 plus 3 plus 2 plus 4.” When asked to indicate how he got his answer, he counted across the top of the square and got 4, continued around the perimeter clockwise, counting 3, 3, and 2. (See Figure 9b.) Finally, he counted the 4 squares in the middle.

In his original counting of squares, DA structured the array as a spiral path, adequately situating it within 2D space. When he later enumerated the squares in the array, he took this previously created abstracted-from-action path as input and restructured it into components consisting of a horizontal 4 units, then a vertical 3, a horizontal 3, a vertical 2, and finally the 2-by-2 group of 4. The direction changes in this path served as spatial delimiters in its decomposition. By segmenting the path into parts, DA took the first step in restructuring it. The following episode shows how BH took the next step in this type of restructuring.

Example 16. BH (the same second grader as in Examples 2 and 4) was shown a 3-by-4 rectangle and was asked to cover it with square tiles. He counted the squares while he placed them in the rectangle in the order shown in Figure 10a. The interviewer then covered the squares and asked BH to draw how the squares looked in the rectangle. After drawing the perimeter of the rectangle, BH drew the top then the bottom row of squares, both one-by-one from left to right—just as he had placed them. Next he drew the left- and right-side squares, again in the same order that he had placed the tiles. BH completed his picture by drawing a vertical segment upward to extend the right side of the first square in the bottom row into the rectangle’s interior (see Figure 10b).

**Int:** How many squares are in your picture?
**BH:** Fourteen. Wait. [Pointing very quickly to the top and bottom rows separately, then
As was previously shown in Example 2, BH originally evidenced a 1D structuring when he counted around the perimeter then the interior of the arrays. In the present example, he moved away from this path conception, structuring the array into top and bottom rows, partial right and left sides, and interior. He had interiorized his original sequential-path structuring so that he could re-present it, “look down” on it, and reconstitute it as a set of components.

But BH had progressed beyond the mere segmenting of the path into components that we observed with DA. BH had disembedded the components from the original sequential path, and he had compared and situated them within 2D space. Nevertheless, BH’s structuring of the array remained sequence-like, still seemingly tied to the original counting series from which it was abstracted. Indeed, the last component in the sequence of components that formed his structuring was “2 squares in the middle,” but the orientation of this component was incorrect. BH had not completed a proper coordination of the components in two dimensions.

**Moving From Local to Global Structuring**

Partial row-column structuring involves local rather than global structuring. That is, when BH structured the array into top, bottom, sides, and interior components, he had organized only parts of the array. He did not coordinate his structuring of the squares or components to obtain a uniform organization that he could apply throughout the array. How is it that students progress from such partial structuring to a global row- or column-structuring of arrays? The following example is suggestive. In it AM predicts how many squares will cover a 4-by-3 rectangle, having been shown only that 4 squares fit across the top and 3 fit down the left side.

**Example 17. (See Figure 11i.)**

AM: [Making her original prediction for an unmarked 4-by-3 rectangle; see Figure 11a] There’s 4 here [top row] and 4 here [bottom row] plus 2 here [one on the left side, one on the right], and that equals 10. [Pointing to what would be the 2 interior squares] But I’m not sure if there’s 2 in the middle or 1 in the middle, because of the size of them.
Initially AM, similarly to BH in Example 16, structured the array as a set of disjoint components—top, bottom, sides, and an amorphous interior—situated in 2D space. She was able to structure the middle of the rectangle as 2 squares, and thus its middle row as 4, only when she saw the 2 middle squares on the sides not as separate, but as part of the right and left columns. This coordinating action enabled her to infer the equivalence of the first and second rows because it vertically aligned their constituent squares and set up a one-to-one correspondence between them. The coordinating action thus enabled AM to progress to a uniform global structuring of the array into three equivalent rows. Moreover, AM seemed
to have interiorized her newly created notion of row-equivalence. Indeed, when she inadvertently turned the rectangle while covering it, she still counted by rows. But she was not simply recalling the rows from her previous counting, because now she correctly counted 4 rows of 3 instead of 3 rows of 4.

On the next problem, AM returned to a prediction based on the top-bottom-right-left structuring that she had used initially with the first problem. However, after drawing the 4-by-5 array correctly, she counted rows of 4:

Example 18. (See Figure 1b.)

AM: Four and 4 equals 8 [sweeping across the bottom 2 rows; see Figure 11e], plus 4 equals [tapping on individual squares] 12, plus 4 equals [again tapping on individual squares] 16, plus 4 equals 21 [not tapping or counting squares].

For the next rectangle, after first counting around the perimeter by ones, AM hesitated when she became confused with the interior. She then returned to organizing her counting by columns of 4:

Example 19. (See Figure 1a.)


In Example 19, AM seemed to recognize in the midst of her initial count that column structuring could help her account for the missing interior squares. This recognition suggests that AM had abstracted a row-by-column structure to a sufficient degree that she could anticipate its usefulness.

In the following episode, PT uses a similar but more dynamic type of coordinating action to structure an array.

Example 20. (See Figure 1p.)

PT: [Counting squares in the bottom row as in Figure 12a] 1, 2, 3, 4, 5, because 1, 2, 3, 4, 5 [pointing to the predrawn squares as in 12b]. Because when you are going up, it is still 5. Just bring them down to make one row [motioning to the bottom row with his fingers]. Bring all the blocks down that are higher, bring them down. It would be the same as one straight line.

Int: Can you predict how many altogether?

PT: Five, 10, 15, 20 [pointing to successive rows as in Figure 12c].
This episode is an excellent example of how students use visual imagery to coordinate and structure parts of an array. PT inferred that there were 5 squares in the bottom row because he could imagine how the 5 predrawn squares could be “brought down” to the bottom row. When PT visualized moving the predrawn squares downward to establish the horizontal positioning of squares in the bottom row, his coordinating action structured the array into two coordinated orthogonal dimensions. It enabled him to see not only that there were 5 squares in each row but also that the rows, if stacked vertically, could compose the whole rectangular array.

In the following example, JH explicitly expresses a logic based on the same type of coordinating actions we have attributed to AM and PT.

Example 21. (See Figure 1p.)

**JH:** I’ve got equal amounts [motioning down the right column, but referring to the rows], so I can count under here; 1, 2, 3, 4, 5 [moving left to right to point to squares in row 3]. Five, 10, 15, 20 [sweeping a finger along successive rows, top to bottom].

**Int:** So it’s not possible that 3 squares would fit here [pointing to the space that would be occupied by the two rightmost squares in row 3]?

**JH:** There’s 2 of them. ‘Cause one right there and one right there [pointing to the above 2 predrawn squares in the upper right].

JH’s remarks suggest that his global row-by-column structuring of the array included interrelationships among squares that were based on spatial coordinations similar to those used by PT. These spatial coordinations enabled JH to infer positions of missing squares from squares that were already drawn. In the next episode, JD uses a similar logic to correctly restructure his drawing of an array.

Example 22. (See Figure 1o.)

**JD:** There’s 1, 2, 3, 4, 5, 6 right down there [pointing to tiles in the first column], 12 [pointing down the second column]; 12 plus 12 is 24 [pointing to the third column], 24 and 6 is (pause) 30 [pointing to the fourth column]. So, I’ll say about 30 in there.

**Int:** Can you draw where you think the squares would be?

![Figure 13. Work of JD.](image)
Students’ Spatial Structuring

[JD draws segments as shown in Figure 13a, getting most intersection points correct, except for segments 1 through 5.]

**JD:** [After drawing segment X] Wait, oh man. [He erases segment 1 and continues segment X through the first column, making segment Y; see Figure 13b.] [While he draws segment Y] This goes right there. So this goes right here [while he extends segment Z across column 2 and then column 1]. [He then erases segments 3 and 5 in Figure 13a.]

Although JD structured the array in terms of equivalent columns, he had difficulty drawing the array correctly. He seemed to visually estimate the position of the squares in column 1, drawing 7 squares, even though he had earlier stated that there were 6. However, when he continued drawing, he noticed the inconsistencies in his drawing because he coordinated the sides of squares that were in different locations, enabling him to maintain the equivalence of the columns. As the next episode illustrates, this type of spatial coordination was missing from CS’s structuring of arrays.

**Example 23.** (See Figure 1p.) CS counts while she draws squares. When she gets to square 4 in row 1, she erases and redraws the right side of square 4, changing it from segment A to segment B, as shown in Figure 14a.

**Int:** Why are you erasing that?

**CS:** Because that [points to the erased right-side segment A] wouldn’t fill up all the space [points to the space between the right side of square 4 and the left side of square 5]: 1, 2, 3, 4, 5 [pointing to the 5 squares in row 1].

[CS draws and counts squares 6, 7, and 8, then pauses because she is not sure how to draw the next square. She completes square 9 as shown in Figure 14b.]

**CS:** Nine, [pointing to the predrawn square] 10, [drawing square 11 in the third row] 11, [pointing to squares and drawing square 15] 12, 13, 14, 15. [CS pauses then counts squares by pointing to them as in Figure 14c] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 [drawing the right side of square 11], 12, 13, 14, 15, 16 [see Figure 14c].

![Figure 14. Work of CS.](image)

Although CS seemed to make some attempt to draw the second row of squares under the first row that she had drawn, she did not coordinate her drawing activity with the predrawn squares. In fact, her drawing of square 15 in Figure 14b created a square to the right of square 10, which CS acknowledged by drawing its right side and counting in a recount. Because her spatial coordination of
squares was local instead of global, she did not notice that this action made her rows nonequivalent.

**Row-by-Column Structuring**

To progress to the final levels of sophistication in structuring 2D rectangular arrays, students must (a) abstract a row (or column) as a composite unit that can be used to compose the whole array and (b) properly coordinate this row-composite-unit with the elements of an orthogonal column so that the row can be distributed over the elements of the column.

*Composing the whole array out of rows-as-composites.* In Examples 5 through 10, we described students’ construction of whole arrays out of copies of rows-as-composites. The following examples illustrate the work of a student who had not reached this level. MS saw the rows or columns as equivalent, but she was unable to compose whole arrays out of these composites.

Example 24. (See Figure 1n.) MS counts one-by-one, left to right across successive rows, getting 21.

**MS:** I counted by the lines and I thought there would be 3 in each like with the lines going across. I thought there would be 3 in each line . . . 3 in each row.

[MS draws the squares as in Figure 15a, getting 3 in each row. However, after referring to the “2 big ones” in the lower right portion of her drawing, she counts and points as in Figure 15b, getting a new prediction of 23.]

Although MS saw 3 squares in a row for her prediction, and even drew 3 squares in each row, she had not globally structured the array as composed of composites of 3.

Example 25. (See Figure 1p.) For her prediction, MS wrote numbers on squares and counted as shown in Figure 15c. When asked how she knew where the squares would be, she stated, “There would be 4 in each row going down,” while she
pointed to several columns. However, when asked to draw the squares and make another prediction, MS drew and counted as shown in Figure 15d.

MS’s solution to the problem presented in Figure 1p further illustrates that, although she saw equivalent rows or columns in arrays, she was unable to utilize these as composites in a global structuring of the arrays. In both her drawing and enumeration, MS was unable to globally structure an array into equivalent columns or rows because she had not yet interiorized structuring an array as a set of row- or column-composites. In fact, it is this interiorization that enables an individual to impose such structuring on an incomplete array during drawing and enumeration (rather than merely to note row equivalence afterward).

*Coordinating a row-as-composite with the elements of the companion orthogonal column.* To construct a global row-by-column structuring of a 2D array, the student must take a row-as-composite and properly distribute it over the elements of a column. That is, the student must activate (or at least symbolize) the row-as-composite for each element in the column. Accomplishing this distribution process is difficult because it requires a reconstitution of the elements of a column, disembedding them from the column and taking them as indicators of successive rows. To increase our understanding of the process of constructing a row-by-column structuring, we examine a student moving from partial row-by-column structuring to genuine row-by-column structuring.

Example 26. (See Figure 1p.) In the left column (see Figure 16a), BH pointed to and counted squares 1 and 2, then used his fingers to measure off and count squares 3 and 4. He then immediately pointed to the right column and said 4. While he pointed to where the squares fit in the bottom row, BH counted 1 through 5, then immediately pointed to the top row and said 5.

*Int:* Where did you get 5?

*Figure 16. Work of BH.*
BH: Since this [the diagonal of predrawn squares] is up, I just have to go 1, this much is down [sweeping his finger down as in A in Figure 16b], that much is down [as in B], that much is down [as in C], and that much is down [as in D]. So that’s 5 on the bottom and 5 on the top. And 1, 2, 3, 4 [sweeping his finger right to left along rows 2, 3, and 4 as shown in Figure 16c]. And 4 on the sides [pointing to the left side].

BH’s sweeping of columns and rows is especially significant because these actions enabled him to coordinate the positions of missing squares with those of predrawn squares in other rows and columns. Indeed, BH himself seemed to recognize the value of this coordinating action because he used it to verify his count of squares in the left column.

Int: How many altogether?
BH: Four on the sides [placing a hand on each side], which equals 8. The top and the bottom and the sides are 18. Now the middle is 19, 20, 21, 22, 23, 24 [while he correctly points to the locations of the squares in the interior].

[After correctly drawing the squares by extending the predrawn lines to the edges, BH counts both the top and bottom rows 1, 2, 3, 4, 5; and both the right and left sides 1, 2, 3, 4.]

BH: So that’s 18 [motioning around the perimeter].

[BH then counts on from 18 one-by-one for the interior 6 squares, getting a total of 24. Because he got the same answer for both his pre- and post-drawing predictions, he thinks 24 is the correct answer.]

For both his original and drawing predictions, BH employed the top-bottom-side-middle structuring that he had used on earlier problems, except that this time he double-counted the corner squares. The quickness with which BH performed these enumerations suggests that his actions were taking on the character of a set procedure—an accepted and partially automatic way of performing the tasks.

BH: [After covering the rectangle with square tiles, BH counts the squares by ones as shown in Figure 16d, then looks puzzled.] Count again.

[BH recounts the squares the same as on his first count except that he tallies square 13 again at the end as 23. He is confused.]

Int: How many across here [separating the first row of squares from the bottom 3 rows just a bit and returning it]?
BH: Five.

Int: How many across here [sweeping his hand along row 2]?
BH: Five (immediately).

Int: How many across here [sweeping his hand along row 3]?
BH: Five (immediately).

Int: How many across here [sweeping his hand along row 4]?
BH: Five (immediately); 5, 10, 15, 20 (excitedly).

[BH now counts the first row by ones—1, 2, 3, 4, 5. He does the same for the second row. He then does the same for the first and second rows again, followed by doing it for the third and fourth rows. BH then counts the rows by 5s two times, then laughs.]

Int: Which do you think is right?
BH: Twenty.

BH achieved a major insight here, finally creating a row-by-column structuring of the array. There is no way to know for sure, but we conjecture that the
sweeping motions that BH used to derive the location of perimeter squares from the locations of interior predrawn squares played a major role in this restructuring. These coordinating actions “lined up” the squares in the rows and columns and became the critical organizational operations that enabled him to see that the rows were equivalent and that the top row could be swept downward and iterated to compose the whole array. We further conjecture that this reorganization did not take place earlier, immediately after the sweeping actions, because BH was still attempting to implement his partially automatic top-bottom-sides-middle enumeration scheme. It was not until BH’s theory was perturbed by his count of tiles and the interviewer subsequently focused his attention on successive rows that these coordinating actions enabled him to restructure the array. BH’s performance on the next problem strongly suggests that he had interiorized a row-by-column structuring.

Example 27. (See Figure 1q.) BH immediately gets 7 squares for both the left and right sides, and 5 for the top and bottom, totaling 24. He then counts by 4s to get an additional 16 for the middle, getting a grand total of 40. He then starts checking with tiles, making the left column first, then the top row, then putting 3 more in the second row, then stops (see Figure 16e).

BH: One, 2, 3, 4, 5 [while he points to the tiles in the top row]. I’m just going to leave it the way it is now; 5, 10, 15, 20, 25, 30, 35 [pointing to successive tiles in the left column, top to bottom].

Int: How did you think of doing it like that?

BH: Because when I looked at the top row, I remembered about the last one I did, how it went 5 [motioning quickly across the top two rows], and I thought whatever way it goes this way, I could just count by numbers like that.

BH started this problem using his top-bottom-sides-middle enumeration scheme. But after he had placed some tiles on the rectangle, he changed his mind. He had sufficiently abstracted the row-by-column structuring he had used on the previous problem so that he could anticipate its applicability and implement it in this new situation. BH did not even need to see the whole array; he needed only enough perceptual material to enable him to perform his row iteration. It may have been significant that the array in this episode had rows of 5, just like the previous array.

DISCUSSION

Overview of the Progression to Row-by-Column Structuring

We have described three levels of sophistication in students’ structuring of 2D arrays of squares. In Level 1, students structure arrays as one-dimensional paths. They follow these paths as if they are traveling along a road and have no awareness of their surroundings, as if in a tunnel. As students become capable of “stepping outside of and looking down on” these paths, they decompose the paths into
components, disembed these components from the sequential organization of a 1D path, and attempt to situate the components in 2D space. This reflective decomposition requires students to abstract their path-creating movements so that they can reflect on, locate, and coordinate the movements within a 2D frame of reference. In Example 4, for instance, the simultaneous inward movement of fingers that BH used to indicate the equivalence and summing of the top and bottom rows of the array suggests that he had organized these rows as opposite and parallel within 2D space. It is the situating of path components in 2D space that enables students to perform the partial row- or column-structuring of Level 2.

The transition to Level 3A requires the student to conceptualize the array as filled by copies of row or column composites. Significantly, this transition seems to have the character of a paradigm shift. It requires the student to abandon a local focus on parts of an array in favor of creating a uniform global structuring of the whole array into rows or columns. We observed the coordinating actions that made this global structuring possible in Examples 20, 21, and 26. Moreover, with AM in Example 17, we actually saw her shift from locally organizing components in 2D space to globally structuring an array in terms of its rows. This shift occurred as AM performed mental actions that coordinated squares within rows and columns in a way that enabled her to see the array as a set of congruent rows.

The transition to Level 3B involves a curtailment of the explicit construction of rows-as-composites within the iteration process. Instead of re-presenting the square-by-square construction of each row composite, students take the squares in a column as indicators of row composites. Functioning as symbolic “pointers,” these squares indicate row composites without requiring re-presentation of the squares within those rows (cf. von Glasersfeld, 1995). It is, in fact, the use of this indicating function that implements the distribution of one composite over the elements of the other as required in a multiplicative scheme (cf. Steffe, 1992).

Finally, movement to row-by-column structuring without perceptual material, Level 3C, requires that students interiorize the entire process of indexed iteration of a row-as-composite. This operationalizes the row-by-column structuring scheme, enabling students to see the whole plan of the scheme from the start and to utilize it in the absence of the original perceptual material (Piaget, Inhelder, & Szeminska, 1960). We saw examples of this operationalization in Examples 11 and 13. Moreover, interiorization is precisely the mechanism that provides the mental material out of which students can construct a mental model for interpreting verbal information such as that presented in Figure 1q.

Perturbations and restructuring. Students’ progression to more sophisticated structurings of arrays often resulted from perturbations caused by recognition of inadequacies in current structurings. For instance, when students realized that they had miscounted squares because they got different answers on two or more enumeration attempts, they reflected on the structuring that underlay their counting in an attempt to avoid retracing or omitting squares. Other times, students
restructured arrays as a result of perturbations that arose when they attempted to 
draw the arrays; drawings provided additional perceptual material that made 
shortcomings of structurings more apparent. Restructuring also occurred when 
interviewer probes about student actions caused students to further reflect on and 
subsequently detect inadequacies in their structurings.

The Structuring Process

CJ, a third grader, was attempting to count square tiles she had correctly placed 
on the rectangle in Figure 1p (Arnoff, 1996). She counted by 2s several times, 
giving different answers and following different paths while she pointed to pairs 
of squares, finally counting as shown in Figure 17a.

Example 28.

*CJ:* This is too confusing. [She stares at the tiles for 20 seconds] I get 20.
*Int:* How did you get it?
*CJ:* I counted by 4s [as shown in Figure 17b].
*Int:* What made you start counting by 4s?
*CJ:* Because, they're just in a row of 4, so I thought why not count by 4s.

We contend that CJ derived her column structuring, not by operating on the 
squares per se, but by operating on her previous acts of counting, the last of 
which had, at least implicitly, a column structure. These acts, through the 
process of abstraction, became objects that could be reflected on and, through 
the further process of reflective abstraction, organized into a new structure 
consisting of column-composites of 4.

This episode, along with many of our previous examples, supports our contention 
that students’ spatial structurings of arrays come as a result of their organizing 
actions (motor and perceptual) on the sets of squares. That is, students create spa-
tial structures for sets of objects through the mental actions they perform on the 
objects. They do not “read off” these structures from objects, but instead, employ a 
process of “constructive structurization” that enriches objects with non-perceptual
content (Piaget & Inhelder, 1969). This process actively establishes interrelationships between objects and is based on the gradual coordination of the individual’s physical and attentional actions (Kosslyn, 1988; Piaget & Inhelder, 1967). Structuring extracts the sequence of coordinating actions used to organize a set of objects to create an entity that itself can be taken as content in further operations such as enumeration. In our study of 3D arrays (Battista & Clements, 1996), we were able to see only the consequences of a student’s ability or inability to perform such coordinating actions. In the present study, we actually observed these actions, resulting in a much better understanding of their nature.

CONCLUSION AND IMPLICATIONS

In the traditional view of learning, it is assumed that row-by-column structuring resides in 2D rectangular arrays of squares and can be automatically apprehended by all. However, as we have seen in the present study, and consistent with a constructivist view of the operation of the mind, such structuring is not “in” the arrays—it must be personally constructed by each individual. Consequently, traditional instructional treatments of multiplication and area need to be rethought. If students do not see a row-by-column structure in these arrays, how can using multiplication to enumerate the objects in the arrays, much less using area formulas, make sense to them?

Taking a broader view, we suggest that structuring 2D and 3D space is the foundation for geometric and visual thinking. All of geometry is, in essence, a way of structuring space and studying the consequences of that structuring. We structure space when we organize it by arrays or coordinate systems. We structure space when we conceptualize it in terms of specific shapes (such as lines, angles, polygons, polyhedra) or in terms of geometric transformations. Studying the processes by which students structure space offers us a new and powerful perspective on investigating children’s construction of geometric and spatial ideas.

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