Prerequisites

P

P.1 Review of Real Numbers and Their Properties

P.2 Exponents and Radicals

P.3 Polynomials and Special Products

P.4 Factoring Polynomials

P.5 Rational Expressions

P.6 Errors and the Algebra of Calculus

P.7 The Rectangular Coordinate System and Graphs

An expanded version of Sections P.1–P.4 is available on the text-specific website at *college.hmco.com* and on the *Interactive* version of this text. This expanded version contains the following sections: Operations with Real Numbers; Properties of Real Numbers; Algebraic Expressions; Operations with Polynomials; Factoring Polynomials; Factoring Trinomials.

You can use the Cartesian plane to organize real-life data such as the number of pieces of mail handled by the U.S. Postal Service each year.



Mary Kate Denny/PhotoEdit

SELECTED APPLICATIONS

Prealgebra concepts have many real-life applications. The applications listed below represent a small sample of the applications in this chapter.

- Budget Variance, Exercises 61–64, page 10
- Period of a Pendulum, Exercise 105, page 23
- Stopping Distance, Exercise 108, page 32
- Interactive Money Management, Exercise 84, page 50
- Meteorology, Exercise 22, page 65
- Data Analysis: Mail, Exercise 65, page 67
- Engineering, Exercise 42, page 71
- Astronomy, Exercise 8, page 76

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P.1 Review of Real Numbers and Their Properties

What you should learn

- Represent and classify real numbers.
- Order real numbers and use inequalities.
- Find the absolute values of real numbers and find the distance between two real numbers.
- Evaluate algebraic expressions.
- Use the basic rules and properties of algebra.

Why you should learn it

Real numbers are used to represent many real-life quantities. For example, in Exercise 65 on page 10, you will use real numbers to represent the federal deficit.

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain additional resources related to the concepts discussed in this chapter.

Real Numbers

Real numbers are used in everyday life to describe quantities such as age, miles per gallon, and population. Real numbers are represented by symbols such as

$$-5,\ 9,\ 0,\ \frac{4}{3},\ 0.666\ldots,\ 28.21,\ \sqrt{2},\ \pi,\ \text{and}\ \sqrt[3]{-32}.$$

Here are some important **subsets** (each member of subset B is also a member of set A) of the real numbers. The three dots, called *ellipsis points*, indicate that the pattern continues indefinitely.

$\{1, 2, 3, 4, \ldots\}$ Set of natural numbers

$\{0, 1, 2, 3, 4, \ldots\}$ Set of whole numbers

$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ Set of integers

A real number is **rational** if it can be written as the ratio p/q of two integers, where $q \neq 0$. For instance, the numbers

$$\frac{1}{3} = 0.3333\ldots = 0.\overline{3},\ \frac{1}{8} = 0.125,\ \text{and}\ \frac{125}{111} = 1.126126\ldots = 1.\overline{126}$$

are rational. The decimal representation of a rational number either repeats $\left(\text{as in } \frac{173}{55} = 3.1\overline{45}\right)$ or terminates $\left(\text{as in } \frac{1}{2} = 0.5\right)$. A real number that cannot be written as the ratio of two integers is called **irrational.** Irrational numbers have infinite nonrepeating decimal representations. For instance, the numbers

$$\sqrt{2} = 1.4142135\ldots \approx 1.41 \quad \text{and} \quad \pi = 3.1415926\ldots \approx 3.14$$

are irrational. (The symbol $\approx$ means "is approximately equal to.") Figure P.1 shows subsets of real numbers and their relationships to each other.

Real numbers are represented graphically by a **real number line.** The point 0 on the real number line is the **origin.** Numbers to the right of 0 are positive, and numbers to the left of 0 are negative, as shown in Figure P.2. The term **nonnegative** describes a number that is either positive or zero.

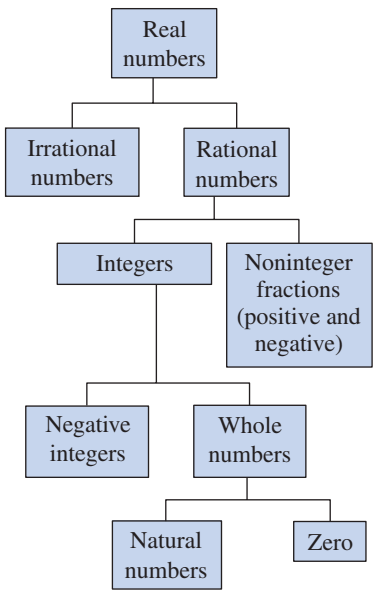


FIGURE P.1 *Subsets of real numbers*

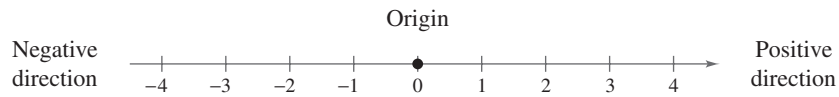
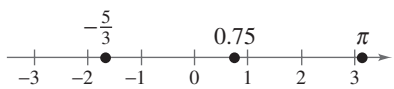
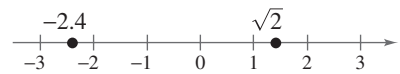


FIGURE P.2 *The real number line*

As illustrated in Figure P.3, there is a *one-to-one correspondence* between real numbers and points on the real number line.



Every real number corresponds to exactly one point on the real number line.



Every point on the real number line corresponds to exactly one real number.

FIGURE P.3 *One-to-one*

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Ordering Real Numbers

One important property of real numbers is that they are *ordered*.

Definition of Order on the Real Number Line

If a and b are real numbers, a is less than b if $b - a$ is positive. The **order** of a and b is denoted by the **inequality** $a < b$. This relationship can also be described by saying that b is *greater than* a and writing $b > a$. The inequality $a \leq b$ means that a is *less than or equal to* b, and the inequality $b \geq a$ means that b is *greater than or equal to* a. The symbols $<$, $>$, $\leq$, and $\geq$ are *inequality symbols*.

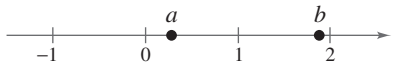


FIGURE P.4 *$a < b$ if and only if a lies to the left of b.*

Geometrically, this definition implies that $a < b$ if and only if a lies to the *left* of b on the real number line, as shown in Figure P.4.

Example 1 **Interpreting Inequalities**

Describe the subset of real numbers represented by each inequality.

a. $x \leq 2$ **b.** $-2 \leq x < 3$

Solution

a. The inequality $x \leq 2$ denotes all real numbers less than or equal to 2, as shown in Figure P.5.

b. The inequality $-2 \leq x < 3$ means that $x \geq -2$ *and* $x < 3$. This "double inequality" denotes all real numbers between -2 and 3, including -2 but not including 3, as shown in Figure P.6.

CHECKPOINT Now try Exercise 19.

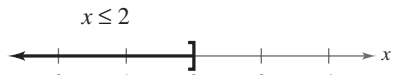


FIGURE P.5

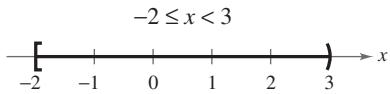
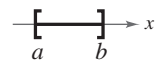
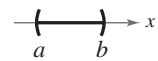
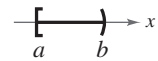
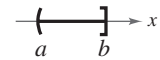


FIGURE P.6

Inequalities can be used to describe subsets of real numbers called **intervals.** In the bounded intervals below, the real numbers a and b are the **endpoints** of each interval. The endpoints of a closed interval are included in the interval, whereas the endpoints of an open interval are not included in the interval.

Bounded Intervals on the Real Number Line

Notation	*Interval Type*	*Inequality*	*Graph*
$[a, b]$	Closed	$a \leq x \leq b$	
(a, b)	Open	$a < x < b$	
$[a, b)$		$a \leq x < b$	
$(a, b]$		$a < x \leq b$	

STUDY TIP

The reason that the four types of intervals at the right are called *bounded* is that each has a finite length. An interval that does not have a finite length is *unbounded* (see page 4).

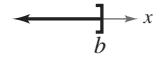
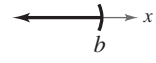
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STUDY TIP

Note that whenever you write intervals containing ∞ or $-\infty$, you always use a parenthesis and never a bracket. This is because these symbols are never an endpoint of an interval and therefore not included in the interval.

The symbols ∞, **positive infinity,** and $-\infty$, **negative infinity,** do not represent real numbers. They are simply convenient symbols used to describe the unboundedness of an interval such as $(1, \infty)$ or $(-\infty, 3]$.

Unbounded Intervals on the Real Number Line

Notation	*Interval Type*	*Inequality*	*Graph*
$[a, \infty)$		$x \geq a$	
(a, ∞)	Open	$x > a$	
$(-\infty, b]$		$x \leq b$	
$(-\infty, b)$	Open	$x < b$	
$(-\infty, \infty)$	Entire real line	$-\infty < x < \infty$	

Example 2 Using Inequalities to Represent Intervals

Use inequality notation to describe each of the following.

a. c is at most 2. **b.** m is at least -3.

c. All x in the interval $(-3, 5]$

Solution

a. The statement "c is at most 2" can be represented by $c \leq 2$.

b. The statement "m is at least -3" can be represented by $m \geq -3$.

c. "All x in the interval $(-3, 5]$" can be represented by $-3 < x \leq 5$.

CHECKPOINT Now try Exercise 31.

Example 3 Interpreting Intervals

Give a verbal description of each interval.

a. $(-1, 0)$ **b.** $[2, \infty)$ **c.** $(-\infty, 0)$

Solution

a. This interval consists of all real numbers that are greater than -1 and less than 0.

b. This interval consists of all real numbers that are greater than or equal to 2.

c. This interval consists of all negative real numbers.

CHECKPOINT Now try Exercise 29.

The **Law of Trichotomy** states that for any two real numbers a and b, *precisely* one of three relationships is possible:

$$a = b, \quad a < b, \quad \text{or} \quad a > b.$$ Law of Trichotomy

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Exploration

Absolute value expressions can be evaluated on a graphing utility. When an expression such as $|3 - 8|$ is evaluated, parentheses should surround the expression, as shown below.



Evaluate each expression. What can you conclude?

a. $|6|$ **b.** $|-1|$
c. $|5 - 2|$ **d.** $|2 - 5|$

Absolute Value and Distance

The **absolute value** of a real number is its *magnitude*, or the distance between the origin and the point representing the real number on the real number line.

Definition of Absolute Value

If a is a real number, then the absolute value of a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}.$$

Notice in this definition that the absolute value of a real number is never negative. For instance, if $a = -5$, then $|-5| = -(-5) = 5$. The absolute value of a real number is either positive or zero. Moreover, 0 is the only real number whose absolute value is 0. So, $|0| = 0$.

Example 4 **Evaluating the Absolute Value of a Number**

Evaluate $\dfrac{|x|}{x}$ for (a) $x > 0$ and (b) $x < 0$.

Solution

a. If $x > 0$, then $|x| = x$ and $\dfrac{|x|}{x} = \dfrac{x}{x} = 1$.

b. If $x < 0$, then $|x| = -x$ and $\dfrac{|x|}{x} = \dfrac{-x}{x} = -1$.

CHECKPOINT Now try Exercise 47.

Properties of Absolute Values

1. $|a| \geq 0$ **2.** $|-a| = |a|$

3. $|ab| = |a||b|$ **4.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, \quad b \neq 0$

Absolute value can be used to define the distance between two points on the real number line. For instance, the distance between -3 and 4 is

$$\begin{aligned} |-3 - 4| &= |-7| \\ &= 7 \end{aligned}$$

as shown in Figure P.7.

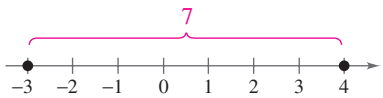


FIGURE P.7 *The distance between* -3 *and* 4 *is* 7.

Distance Between Two Points on the Real Number Line

Let a and b be real numbers. The **distance between a and b** is

$$d(a, b) = |b - a| = |a - b|.$$

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Algebraic Expressions

One characteristic of algebra is the use of letters to represent numbers. The letters are **variables,** and combinations of letters and numbers are **algebraic expressions.** Here are a few examples of algebraic expressions.

$$5x, \qquad 2x - 3, \qquad \frac{4}{x^2 + 2}, \qquad 7x + y$$

Definition of an Algebraic Expression

An **algebraic expression** is a collection of letters **(variables)** and real numbers **(constants)** combined using the operations of addition, subtraction, multiplication, division, and exponentiation.

The **terms** of an algebraic expression are those parts that are separated by *addition*. For example,

$$x^2 - 5x + 8 = x^2 + (-5x) + 8$$

has three terms: x^2 and $-5x$ are the **variable terms** and 8 is the **constant term.** The numerical factor of a variable term is the **coefficient** of the variable term. For instance, the coefficient of $-5x$ is -5, and the coefficient of x^2 is 1.

To **evaluate** an algebraic expression, substitute numerical values for each of the variables in the expression. Here are two examples.

Expression	*Value of Variable*	*Substitute*	*Value of Expression*
$-3x + 5$	$x = 3$	$-3(3) + 5$	$-9 + 5 = -4$
$3x^2 + 2x - 1$	$x = -1$	$3(-1)^2 + 2(-1) - 1$	$3 - 2 - 1 = 0$

When an algebraic expression is evaluated, the **Substitution Principle** is used. It states that "If $a = b$, then a can be replaced by b in any expression involving a." In the first evaluation shown above, for instance, 3 is *substituted* for x in the expression $-3x + 5$.

Basic Rules of Algebra

There are four arithmetic operations with real numbers: *addition*, *multiplication*, *subtraction*, and *division*, denoted by the symbols $+$, $\times$ or $\cdot$, $-$, and $\div$ or /. Of these, addition and multiplication are the two primary operations. Subtraction and division are the inverse operations of addition and multiplication, respectively.

Definitions of Subtraction and Division

Subtraction: Add the opposite.

$$a - b = a + (-b)$$

Division: Multiply by the reciprocal.

If $b \neq 0$, then $a/b = a\left(\frac{1}{b}\right) = \frac{a}{b}$.

In these definitions, $-b$ is the **additive inverse** (or opposite) of b, and $1/b$ is the **multiplicative inverse** (or reciprocal) of b. In the fractional form a/b, a is the **numerator** of the fraction and b is the **denominator.**

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Because the properties of real numbers below are true for variables and algebraic expressions as well as for real numbers, they are often called the **Basic Rules of Algebra.** Try to formulate a verbal description of each property. For instance, the first property states that *the order in which two real numbers are added does not affect their sum.*

Basic Rules of Algebra

Let a, b, and c be real numbers, variables, or algebraic expressions.

Property		*Example*
Commutative Property of Addition:	$a + b = b + a$	$4x + x^2 = x^2 + 4x$
Commutative Property of Multiplication:	$ab = ba$	$(4 - x)x^2 = x^2(4 - x)$
Associative Property of Addition:	$(a + b) + c = a + (b + c)$	$(x + 5) + x^2 = x + (5 + x^2)$
Associative Property of Multiplication:	$(ab)c = a(bc)$	$(2x \cdot 3y)(8) = (2x)(3y \cdot 8)$
Distributive Properties:	$a(b + c) = ab + ac$	$3x(5 + 2x) = 3x \cdot 5 + 3x \cdot 2x$
	$(a + b)c = ac + bc$	$(y + 8)y = y \cdot y + 8 \cdot y$
Additive Identity Property:	$a + 0 = a$	$5y^2 + 0 = 5y^2$
Multiplicative Identity Property:	$a \cdot 1 = a$	$(4x^2)(1) = 4x^2$
Additive Inverse Property:	$a + (-a) = 0$	$5x^3 + (-5x^3) = 0$
Multiplicative Inverse Property:	$a \cdot \frac{1}{a} = 1, \quad a \neq 0$	$(x^2 + 4)\left(\frac{1}{x^2 + 4}\right) = 1$

Because subtraction is defined as "adding the opposite," the Distributive Properties are also true for subtraction. For instance, the "subtraction form" of $a(b + c) = ab + ac$ is $a(b - c) = ab - ac$.

Properties of Negation and Equality

Let a and b be real numbers, variables, or algebraic expressions.

Property	*Example*
1. $(-1)a = -a$	$(-1)7 = -7$
2. $-(-a) = a$	$-(-6) = 6$
3. $(-a)b = -(ab) = a(-b)$	$(-5)3 = -(5 \cdot 3) = 5(-3)$
4. $(-a)(-b) = ab$	$(-2)(-x) = 2x$
5. $-(a + b) = (-a) + (-b)$	$-(x + 8) = (-x) + (-8)$ $= -x - 8$
6. If $a = b$, then $a \pm c = b \pm c$.	$\frac{1}{2} + 3 = 0.5 + 3$
7. If $a = b$, then $ac = bc$.	$4^2 \cdot 2 = 16 \cdot 2$
8. If $a \pm c = b \pm c$, then $a = b$.	$1.4 - 1 = \frac{7}{5} - 1 \Rightarrow 1.4 = \frac{7}{5}$
9. If $ac = bc$ and $c \neq 0$, then $a = b$.	$3x = 3 \cdot 4 \Rightarrow x = 4$

STUDY TIP

Notice the difference between the *opposite of a number* and a *negative number*. If a is already negative, then its opposite, $-a$, is positive. For instance, if $a = -5$, then

$$-a = -(-5) = 5.$$

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STUDY TIP

The "or" in the Zero-Factor Property includes the possibility that either or both factors may be zero. This is an **inclusive or,** and it is the way the word "or" is generally used in mathematics.

Properties of Zero

Let a and b be real numbers, variables, or algebraic expressions.

1. $a + 0 = a$ and $a - 0 = a$
2. $a \cdot 0 = 0$
3. $\frac{0}{a} = 0, \quad a \neq 0$
4. $\frac{a}{0}$ is undefined.
5. **Zero-Factor Property:** If $ab = 0$, then $a = 0$ or $b = 0$.

Properties and Operations of Fractions

Let a, b, c, and d be real numbers, variables, or algebraic expressions such that $b \neq 0$ and $d \neq 0$.

1. **Equivalent Fractions:** $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.
2. **Rules of Signs:** $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ and $\frac{-a}{-b} = \frac{a}{b}$
3. **Generate Equivalent Fractions:** $\frac{a}{b} = \frac{ac}{bc}, \quad c \neq 0$
4. **Add or Subtract with Like Denominators:** $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$
5. **Add or Subtract with Unlike Denominators:** $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$
6. **Multiply Fractions:** $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
7. **Divide Fractions:** $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}, \quad c \neq 0$

STUDY TIP

In Property 1 of fractions, the phrase "if and only if" implies two statements. One statement is: If $a/b = c/d$, then $ad = bc$. The other statement is: If $ad = bc$, where $b \neq 0$ and $d \neq 0$, then $a/b = c/d$.

Example 5 **Properties and Operations of Fractions**

a. Equivalent fractions: $\frac{x}{5} = \frac{3 \cdot x}{3 \cdot 5} = \frac{3x}{15}$ **b.** Divide fractions: $\frac{7}{x} \div \frac{3}{2} = \frac{7}{x} \cdot \frac{2}{3} = \frac{14}{3x}$

c. Add fractions with unlike denominators: $\frac{x}{3} + \frac{2x}{5} = \frac{5 \cdot x + 3 \cdot 2x}{3 \cdot 5} = \frac{11x}{15}$

CHECKPOINT Now try Exercise 103.

If a, b, and c are integers such that $ab = c$, then a and b are **factors** or **divisors** of c. A **prime number** is an integer that has exactly two positive factors — itself and 1 — such as 2, 3, 5, 7, and 11. The numbers 4, 6, 8, 9, and 10 are **composite** because each can be written as the product of two or more prime numbers. The number 1 is neither prime nor composite. The **Fundamental Theorem of Arithmetic** states that every positive integer greater than 1 can be written as the product of prime numbers in precisely one way (disregarding order). For instance, the *prime factorization* of 24 is $24 = 2 \cdot 2 \cdot 2 \cdot 3$.

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P.1 Exercises

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

VOCABULARY CHECK: Fill in the blanks.

1. A real number is _____ if it can be written as the ratio $\frac{p}{q}$ of two integers, where $q \neq 0$.
2. _____ numbers have infinite nonrepeating decimal representations.
3. The distance between a point on the real number line and the origin is the _____ _____ of the real number.
4. A number that can be written as the product of two or more prime numbers is called a _____ number.
5. An integer that has exactly two positive factors, the integer itself and 1, is called a _____ number.
6. An algebraic expression is a collection of letters called _____ and real numbers called _____.
7. The _____ of an algebraic expression are those parts separated by addition.
8. The numerical factor of a variable term is the _____ of the variable term.
9. The _____ _____ states that if $ab = 0$, then $a = 0$ or $b = 0$.

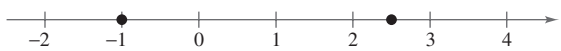
In Exercises 1–6, determine which numbers in the set are (a) natural numbers, (b) whole numbers, (c) integers, (d) rational numbers, and (e) irrational numbers.

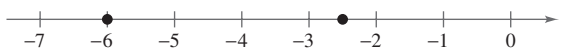
1. $-9, -\frac{7}{2}, 5, \frac{2}{3}, \sqrt{2}, 0, 1, -4, 2, -11$
2. $\sqrt{5}, -7, -\frac{7}{3}, 0, 3.12, \frac{5}{4}, -3, 12, 5$
3. $2.01, 0.666\ldots, -13, 0.010110111\ldots, 1, -6$
4. $2.3030030003\ldots, 0.7575, -4.63, \sqrt{10}, -75, 4$
5. $-\pi, -\frac{1}{3}, \frac{6}{3}, \frac{1}{2}\sqrt{2}, -7.5, -1, 8, -22$
6. $25, -17, -\frac{12}{5}, \sqrt{9}, 3.12, \frac{1}{2}\pi, 7, -11.1, 13$

In Exercises 7–10, use a calculator to find the decimal form of the rational number. If it is a nonterminating decimal, write the repeating pattern.

7. $\frac{5}{8}$
8. $\frac{1}{3}$
9. $\frac{41}{333}$
10. $\frac{6}{11}$

In Exercises 11 and 12, approximate the numbers and place the correct symbol (< or >) between them.

11. 

12. 

In Exercises 13–18, plot the two real numbers on the real number line. Then place the appropriate inequality symbol (< or >) between them.

13. $-4, -8$
14. $-3.5, 1$
15. $\frac{3}{2}, 7$
16. $1, \frac{16}{3}$
17. $\frac{5}{6}, \frac{2}{3}$
18. $-\frac{8}{7}, -\frac{3}{7}$

In Exercises 19–30, (a) give a verbal description of the subset of real numbers represented by the inequality or the interval, (b) sketch the subset on the real number line, and (c) state whether the interval is bounded or unbounded.

19. $x \leq 5$
20. $x \geq -2$
21. $x < 0$
22. $x > 3$
23. $[4, \infty)$
24. $(-\infty, 2)$
25. $-2 < x < 2$
26. $0 \leq x \leq 5$
27. $-1 \leq x < 0$
28. $0 < x \leq 6$
29. $[-2, 5)$
30. $(-1, 2]$

In Exercises 31–38, use inequality notation to describe the set.

31. All x in the interval $(-2, 4]$
32. All y in the interval $[-6, 0)$
33. y is nonnegative.
34. y is no more than 25.
35. t is at least 10 and at most 22.
36. k is less than 5 but no less than -3.
37. The dog's weight W is more than 65 pounds.
38. The annual rate of inflation r is expected to be at least 2.5% but no more than 5%.

In Exercises 39–48, evaluate the expression.

39. $|-10|$
40. $|0|$
41. $|3 - 8|$
42. $|4 - 1|$
43. $|-1| - |-2|$
44. $-3 - |-3|$
45. $\frac{-5}{|-5|}$
46. $-3|-3|$
47. $\frac{|x+2|}{x+2}, \quad x < -2$
48. $\frac{|x-1|}{x-1}, \quad x > 1$

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In Exercises 49–54, place the correct symbol (<, >, or =) between the pair of real numbers.

49. $|-3|$ ☐ $-|-3|$

50. $|-4|$ ☐ $|4|$

51. -5 ☐ $-|5|$

52. $-|-6|$ ☐ $|-6|$

53. $-|-2|$ ☐ $-|2|$

54. $-(-2)$ ☐ -2

In Exercises 55–60, find the distance between *a* and *b*.

55. $a = 126, b = 75$

56. $a = -126, b = -75$

57. $a = -\frac{5}{2}, b = 0$

58. $a = \frac{1}{4}, b = \frac{11}{4}$

59. $a = \frac{16}{5}, b = \frac{112}{75}$

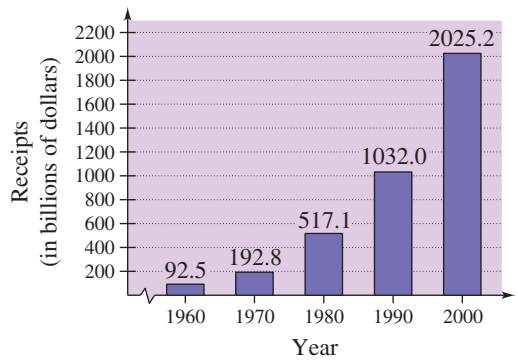
60. $a = 9.34, b = -5.65$

Budget Variance **In Exercises 61–64, the accounting department of a sports drink bottling company is checking to see whether the actual expenses of a department differ from the budgeted expenses by more than $500 or by more than 5%. Fill in the missing parts of the table, and determine whether each actual expense passes the "budget variance test."**

	Budgeted Expense, b	*Actual Expense, a*	$\lvert a - b \rvert$	$0.05b$
61. Wages	$112,700	$113,356		
62. Utilities	$9,400	$9,772		
63. Taxes	$37,640	$37,335		
64. Insurance	$2,575	$2,613		

Model It

65. ***Federal Deficit*** The bar graph shows the federal government receipts (in billions of dollars) for selected years from 1960 through 2000. (Source: U.S. Office of Management and Budget)



(a) Complete the table. (*Hint:* Find |Receipts – Expenditures|.)

Model It (continued)

Year	Expenditures (in billions)	Surplus or deficit (in billions)
1960	$92.2	
1970	$195.6	
1980	$590.9	
1990	$1253.2	
2000	$1788.8	

(b) Use the table in part (a) to construct a bar graph showing the magnitude of the surplus or deficit for each year.

66. ***Veterans*** The table shows the number of living veterans (in thousands) in the United States in 2002 by age group. Construct a circle graph showing the percent of living veterans by age group as a fraction of the total number of living veterans. (Source: Department of Veteran Affairs)

Age group	Number of veterans
Under 35	2213
35–44	3290
45–54	4666
55–64	5665
65 and older	9784

In Exercises 67–72, use absolute value notation to describe the situation.

67. The distance between x and 5 is no more than 3.

68. The distance between x and -10 is at least 6.

69. y is at least six units from 0.

70. y is at most two units from a.

71. While traveling on the Pennsylvania Turnpike, you pass milepost 326 near Valley Forge, then milepost 351 near Philadelphia. How many miles do you travel during that time period?

72. The temperature in Chicago, Illinois was 48° last night at midnight, then 82° at noon today. What was the change in temperature over the 12-hour period?

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In Exercises 73–78, identify the terms. Then identify the coefficients of the variable terms of the expression.

73. $7x + 4$

74. $6x^3 - 5x$

75. $\sqrt{3}x^2 - 8x - 11$

76. $3\sqrt{3}x^2 + 1$

77. $4x^3 + \frac{x}{2} - 5$

78. $3x^4 - \frac{x^2}{4}$

In Exercises 79–84, evaluate the expression for each value of *x*. (If not possible, state the reason.)

	Expression	*Values*	
79.	$4x - 6$	(a) $x = -1$	(b) $x = 0$
80.	$9 - 7x$	(a) $x = -3$	(b) $x = 3$
81.	$x^2 - 3x + 4$	(a) $x = -2$	(b) $x = 2$
82.	$-x^2 + 5x - 4$	(a) $x = -1$	(b) $x = 1$
83.	$\frac{x+1}{x-1}$	(a) $x = 1$	(b) $x = -1$
84.	$\frac{x}{x+2}$	(a) $x = 2$	(b) $x = -2$

In Exercises 85–96, identify the rule(s) of algebra illustrated by the statement.

85. $x + 9 = 9 + x$

86. $2\left(\frac{1}{2}\right) = 1$

87. $\frac{1}{h+6}(h+6) = 1, \quad h \neq -6$

88. $(x + 3) - (x + 3) = 0$

89. $2(x + 3) = 2 \cdot x + 2 \cdot 3$

90. $(z - 2) + 0 = z - 2$

91. $1 \cdot (1 + x) = 1 + x$

92. $(z + 5)x = z \cdot x + 5 \cdot x$

93. $x + (y + 10) = (x + y) + 10$

94. $x(3y) = (x \cdot 3)y = (3x)y$

95. $3(t - 4) = 3 \cdot t - 3 \cdot 4$

96. $\frac{1}{7}(7 \cdot 12) = \left(\frac{1}{7} \cdot 7\right)12 = 1 \cdot 12 = 12$

In Exercises 97–104, perform the operation(s). (Write fractional answers in simplest form.)

97. $\frac{3}{16} + \frac{5}{16}$

98. $\frac{6}{7} - \frac{4}{7}$

99. $\frac{5}{8} - \frac{5}{12} + \frac{1}{6}$

100. $\frac{10}{11} + \frac{6}{33} - \frac{13}{66}$

101. $12 \div \frac{1}{4}$

102. $-\left(6 \cdot \frac{4}{8}\right)$

103. $\frac{2x}{3} - \frac{x}{4}$

104. $\frac{5x}{6} \cdot \frac{2}{9}$

105. (a) Use a calculator to complete the table.

n	1	0.5	0.01	0.0001	0.000001
$5/n$					

(b) Use the result from part (a) to make a conjecture about the value of $5/n$ as n approaches 0.

106. (a) Use a calculator to complete the table.

n	1	10	100	10,000	100,000
$5/n$					

(b) Use the result from part (a) to make a conjecture about the value of $5/n$ as n increases without bound.

Synthesis

True or False? **In Exercises 107 and 108, determine whether the statement is true or false. Justify your answer.**

107. If $a < b$, then $\frac{1}{a} < \frac{1}{b}$, where $a \neq b \neq 0$.

108. Because $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$, then $\frac{c}{a+b} = \frac{c}{a} + \frac{c}{b}$.

109. ***Exploration*** Consider $|u + v|$ and $|u| + |v|$, where $u \neq v \neq 0$.

(a) Are the values of the expressions always equal? If not, under what conditions are they unequal?

(b) If the two expressions are not equal for certain values of u and v, is one of the expressions always greater than the other? Explain.

110. ***Think About It*** Is there a difference between saying that a real number is positive and saying that a real number is nonnegative? Explain.

111. ***Think About It*** Because every even number is divisible by 2, is it possible that there exist any even prime numbers? Explain.

112. ***Writing*** Describe the differences among the sets of natural numbers, whole numbers, integers, rational numbers, and irrational numbers.

In Exercises 113 and 114, use the real numbers *A*, *B*, and *C* shown on the number line. Determine the sign of each expression.



113. (a) $-A$

(b) $B - A$

114. (a) $-C$

(b) $A - C$

115. ***Writing*** Can it ever be true that $|a| = -a$ for a real number a? Explain.

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P.2 Exponents and Radicals

What you should learn

- Use properties of exponents.
- Use scientific notation to represent real numbers.
- Use properties of radicals.
- Simplify and combine radicals.
- Rationalize denominators and numerators.
- Use properties of rational exponents.

Why you should learn it

Real numbers and algebraic expressions are often written with exponents and radicals. For instance, in Exercise 107 on page 23, you will use an expression involving rational exponents to find the time required for a funnel to empty for different water heights.

Integer Exponents

Repeated *multiplication* can be written in **exponential form.**

Repeated Multiplication	*Exponential Form*
$a \cdot a \cdot a \cdot a \cdot a$	a^5
$(-4)(-4)(-4)$	$(-4)^3$
$(2x)(2x)(2x)(2x)$	$(2x)^4$

Exponential Notation

If a is a real number and n is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

where n is the **exponent** and a is the **base.** The expression a^n is read "a to the nth **power.**"

An exponent can also be negative. In Property 3 below, be sure you see how to use a negative exponent.

Properties of Exponents

Let a and b be real numbers, variables, or algebraic expressions, and let m and n be integers. (All denominators and bases are nonzero.)

Property	*Example*
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^4 = 3^{2+4} = 3^6 = 729$
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{x^7}{x^4} = x^{7-4} = x^3$
3. $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$	$y^{-4} = \dfrac{1}{y^4} = \left(\dfrac{1}{y}\right)^4$
4. $a^0 = 1, \quad a \neq 0$	$(x^2 + 1)^0 = 1$
5. $(ab)^m = a^m b^m$	$(5x)^3 = 5^3 x^3 = 125x^3$
6. $(a^m)^n = a^{mn}$	$(y^3)^{-4} = y^{3(-4)} = y^{-12} = \dfrac{1}{y^{12}}$
7. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{2}{x}\right)^3 = \dfrac{2^3}{x^3} = \dfrac{8}{x^3}$
8. $\lvert a^2 \rvert = \lvert a \rvert^2 = a^2$	$\lvert (-2)^2 \rvert = \lvert -2 \rvert^2 = (2)^2 = 4$

Technology

You can use a calculator to evaluate exponential expressions. When doing so, it is important to know when to use parentheses because the calculator follows the order of operations. For instance, evaluate $(-2)^4$ as follows

Scientific:

(2 +/−) y^x 4 =

Graphing:

((−) 2) ^ 4 ENTER

The display will be 16. If you omit the parentheses, the display will be −16.

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It is important to recognize the difference between expressions such as $(-2)^4$ and -2^4. In $(-2)^4$, the parentheses indicate that the exponent applies to the negative sign as well as to the 2, but in $-2^4 = -(2^4)$, the exponent applies only to the 2. So, $(-2)^4 = 16$ and $-2^4 = -16$.

The properties of exponents listed on the preceding page apply to *all* integers m and n, not just to positive integers as shown in the examples in this section.

Example 1 Using Properties of Exponents

Use the properties of exponents to simplify each expression.

a. $(-3ab^4)(4ab^{-3})$ **b.** $(2xy^2)^3$ **c.** $3a(-4a^2)^0$ **d.** $\left(\frac{5x^3}{y}\right)^2$

Solution

a. $(-3ab^4)(4ab^{-3}) = (-3)(4)(a)(a)(b^4)(b^{-3}) = -12a^2b$

b. $(2xy^2)^3 = 2^3(x)^3(y^2)^3 = 8x^3y^6$

c. $3a(-4a^2)^0 = 3a(1) = 3a, \quad a \neq 0$

d. $\left(\frac{5x^3}{x}\right)^2 = \frac{5^2(x^3)^2}{y^2} = \frac{25x^6}{y^2}$

CHECKPOINT Now try Exercise 25.

Example 2 Rewriting with Positive Exponents

Rewrite each expression with positive exponents.

a. x^{-1} **b.** $\frac{1}{3x^{-2}}$ **c.** $\frac{12a^3b^{-4}}{4a^{-2}b}$ **d.** $\left(\frac{3x^2}{y}\right)^{-2}$

Solution

a. $x^{-1} = \frac{1}{x}$ Property 3

b. $\frac{1}{3x^{-2}} = \frac{1(x^2)}{3} = \frac{x^2}{3}$ The exponent -2 does not apply to 3.

c. $\frac{12a^3b^{-4}}{4a^{-2}b} = \frac{12a^3 \cdot a^2}{4b \cdot b^4} = \frac{3a^5}{b^5}$ Properties 3 and 1

d. $\left(\frac{3x^2}{y}\right)^{-2} = \frac{3^{-2}(x^2)^{-2}}{y^{-2}}$ Properties 5 and 7

$= \frac{3^{-2}x^{-4}}{y^{-2}}$ Property 6

$= \frac{y^2}{3^2x^4}$ Property 3

$= \frac{y^2}{9x^4}$ Simplify.

CHECKPOINT Now try Exercise 33.

STUDY TIP

Rarely in algebra is there only one way to solve a problem. Don't be concerned if the steps you use to solve a problem are not exactly the same as the steps presented in this text. The important thing is to use steps that you understand and, of course, steps that are justified by the rules of algebra. For instance, you might prefer the following steps for Example 2(d).

$$\left(\frac{3x^2}{y}\right)^{-2} = \left(\frac{y}{3x^2}\right)^2 = \frac{y^2}{9x^4}$$

Note how Property 3 is used in the first step of this solution. The fractional form of this property is

$$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m.$$

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Historical Note
The French mathematician Nicolas Chuquet (ca. 1500) wrote *Triparty en la science des nombres*, in which a form of exponent notation was used. Our expressions $6x^3$ and $10x^2$ were written as $.6.^3$ and $.10.^2$. Zero and negative exponents were also represented, so x^0 would be written as $.1.^0$ and $3x^{-2}$ as $.3.^{2m}$. Chuquet wrote that $.72.^1$ divided by $.8.^3$ is $.9.^{2m}$. That is, $72x \div 8x^3 = 9x^{-2}$.

Scientific Notation

Exponents provide an efficient way of writing and computing with very large (or very small) numbers. For instance, there are about 359 billion billion gallons of water on Earth—that is, 359 followed by 18 zeros.

$$359{,}000{,}000{,}000{,}000{,}000{,}000$$

It is convenient to write such numbers in **scientific notation.** This notation has the form $\pm c \times 10^n$, where $1 \le c < 10$ and n is an integer. So, the number of gallons of water on Earth can be written in scientific notation as

$$3.59 \times 100{,}000{,}000{,}000{,}000{,}000{,}000 = 3.59 \times 10^{20}.$$

The *positive* exponent 20 indicates that the number is *large* (10 or more) and that the decimal point has been moved 20 places. A *negative* exponent indicates that the number is *small* (less than 1). For instance, the mass (in grams) of one electron is approximately

$$9.0 \times 10^{-28} = 0.0000000000000000000000000009.$$

28 decimal places

Example 3 Scientific Notation

Write each number in scientific notation.

a. 0.0000782 **b.** 836,100,000

Solution

a. $0.0000782 = 7.82 \times 10^{-5}$ **b.** $836{,}100{,}000 = 8.361 \times 10^8$

CHECKPOINT Now try Exercise 37.

Example 4 Decimal Notation

Write each number in decimal notation.

a. 9.36×10^{-6} **b.** 1.345×10^2

Solution

a. $9.36 \times 10^{-6} = 0.00000936$ **b.** $1.345 \times 10^2 = 134.5$

CHECKPOINT Now try Exercise 41.

Technology

Most calculators automatically switch to scientific notation when they are showing large (or small) numbers that exceed the display range.

To *enter* numbers in scientific notation, your calculator should have an exponential entry key labeled

EE or EXP.

Consult the user's guide for your calculator for instructions on keystrokes and how numbers in scientific notation are displayed.

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Radicals and Their Properties

A **square root** of a number is one of its two equal factors. For example, 5 is a square root of 25 because 5 is one of the two equal factors of 25. In a similar way, a **cube root** of a number is one of its three equal factors, as in $125 = 5^3$.

Definition of *n*th Root of a Number

Let a and b be real numbers and let $n \geq 2$ be a positive integer. If

$$a = b^n$$

then b is an ***n*th root of *a*.** If $n = 2$, the root is a **square root.** If $n = 3$, the root is a **cube root.**

Some numbers have more than one nth root. For example, both 5 and -5 are square roots of 25. The *principal square root* of 25, written as $\sqrt{25}$, is the positive root, 5. The **principal *n*th root** of a number is defined as follows.

Principal *n*th Root of a Number

Let a be a real number that has at least one nth root. The **principal *n*th root of *a*** is the nth root that has the same sign as a. It is denoted by a **radical symbol**

$$\sqrt[n]{a}.$$ Principal nth root

The positive integer n is the **index** of the radical, and the number a is the **radicand.** If $n = 2$, omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$. (The plural of index is *indices*.)

A common misunderstanding is that the square root sign implies both negative and positive roots. This is not correct. The square root sign implies only a positive root. When a negative root is needed, you must use the negative sign with the square root sign.

Incorrect: ~~$\sqrt{4} = \pm 2$~~ *Correct:* $-\sqrt{4} = -2$ *and* $\sqrt{4} = 2$

Example 5 Evaluating Expressions Involving Radicals

a. $\sqrt{36} = 6$ because $6^2 = 36$.

b. $-\sqrt{36} = -6$ because $-\left(\sqrt{36}\right) = -\left(\sqrt{6^2}\right) = -(6) = -6$.

c. $\sqrt[3]{\dfrac{125}{64}} = \dfrac{5}{4}$ because $\left(\dfrac{5}{4}\right)^3 = \dfrac{5^3}{4^3} = \dfrac{125}{64}$.

d. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.

e. $\sqrt[4]{-81}$ is not a real number because there is no real number that can be raised to the fourth power to produce -81.

CHECKPOINT Now try Exercise 51.

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Here are some generalizations about the nth roots of real numbers.

Generalizations About nth Roots of Real Numbers			
Real Number a	Integer n	Root(s) of a	Example
$a > 0$	$n > 0$, is even.	$\sqrt[n]{a}, -\sqrt[n]{a}$	$\sqrt[4]{81} = 3, -\sqrt[4]{81} = -3$
$a > 0$ or $a < 0$	n is odd.	$\sqrt[n]{a}$	$\sqrt[3]{-8} = -2$
$a < 0$	n is even.	No real roots	$\sqrt{-4}$ is not a real number.
$a = 0$	n is even or odd.	$\sqrt[n]{0} = 0$	$\sqrt[5]{0} = 0$

Integers such as 1, 4, 9, 16, 25, and 36 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

Properties of Radicals

Let a and b be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let m and n be positive integers.

	Property	*Example*
1.	$\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	$\sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$
2.	$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$
3.	$\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}, \quad b \neq 0$	$\dfrac{\sqrt[4]{27}}{\sqrt[4]{9}} = \sqrt[4]{\dfrac{27}{9}} = \sqrt[4]{3}$
4.	$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt{10}} = \sqrt[6]{10}$
5.	$\left(\sqrt[n]{a}\right)^n = a$	$\left(\sqrt{3}\right)^2 = 3$
6.	For n even, $\sqrt[n]{a^n} = \lvert a \rvert$.	$\sqrt{(-12)^2} = \lvert -12 \rvert = 12$
	For n odd, $\sqrt[n]{a^n} = a$.	$\sqrt[3]{(-12)^3} = -12$

A common special case of Property 6 is $\sqrt{a^2} = |a|$.

Example 6 Using Properties of Radicals

Use the properties of radicals to simplify each expression.

a. $\sqrt{8} \cdot \sqrt{2}$ **b.** $\left(\sqrt[3]{5}\right)^3$ **c.** $\sqrt[3]{x^3}$ **d.** $\sqrt[6]{y^6}$

Solution

a. $\sqrt{8} \cdot \sqrt{2} = \sqrt{8 \cdot 2} = \sqrt{16} = 4$

b. $\left(\sqrt[3]{5}\right)^3 = 5$

c. $\sqrt[3]{x^3} = x$

d. $\sqrt[6]{y^6} = |y|$

CHECKPOINT Now try Exercise 61.

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Simplifying Radicals

An expression involving radicals is in **simplest form** when the following conditions are satisfied.

1. All possible factors have been removed from the radical.
2. All fractions have radical-free denominators (accomplished by a process called *rationalizing the denominator*).
3. The index of the radical is reduced.

To simplify a radical, factor the radicand into factors whose exponents are multiples of the index. The roots of these factors are written outside the radical, and the "leftover" factors make up the new radicand.

STUDY TIP

When you simplify a radical, it is important that both expressions are defined for the same values of the variable. For instance, in Example 7(b), $\sqrt{75x^3}$ and $5x\sqrt{3x}$ are both defined only for nonnegative values of x. Similarly, in Example 7(c), $\sqrt[4]{(5x)^4}$ and $5|x|$ are both defined for all real values of x.

Example 7 Simplifying Even Roots

Perfect 4th power / Leftover factor

a. $\sqrt[4]{48} = \sqrt[4]{16 \cdot 3} = \sqrt[4]{2^4 \cdot 3} = 2\sqrt[4]{3}$

Perfect square / Leftover factor

b. $\sqrt{75x^3} = \sqrt{25x^2 \cdot 3x}$ Find largest square factor.

$= \sqrt{(5x)^2 \cdot 3x}$

$= 5x\sqrt{3x}$ Find root of perfect square.

c. $\sqrt[4]{(5x)^4} = |5x| = 5|x|$

CHECKPOINT Now try Exercise 63(a).

Example 8 Simplifying Odd Roots

Perfect cube / Leftover factor

a. $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3} = \sqrt[3]{2^3 \cdot 3} = 2\sqrt[3]{3}$

Perfect cube / Leftover factor

b. $\sqrt[3]{24a^4} = \sqrt[3]{8a^3 \cdot 3a}$ Find largest cube factor.

$= \sqrt[3]{(2a)^3 \cdot 3a}$

$= 2a\sqrt[3]{3a}$ Find root of perfect cube.

c. $\sqrt[3]{-40x^6} = \sqrt[3]{(-8x^6) \cdot 5}$ Find largest cube factor.

$= \sqrt[3]{(-2x^2)^3 \cdot 5}$

$= -2x^2\sqrt[3]{5}$ Find root of perfect cube.

CHECKPOINT Now try Exercise 63(b).

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Radical expressions can be combined (added or subtracted) if they are **like radicals**—that is, if they have the same index and radicand. For instance, $\sqrt{2}$, $3\sqrt{2}$, and $\frac{1}{2}\sqrt{2}$ are like radicals, but $\sqrt{3}$ and $\sqrt{2}$ are unlike radicals. To determine whether two radicals can be combined, you should first simplify each radical.

Example 9 Combining Radicals

a. $2\sqrt{48} - 3\sqrt{27} = 2\sqrt{16 \cdot 3} - 3\sqrt{9 \cdot 3}$ Find square factors.

$= 8\sqrt{3} - 9\sqrt{3}$ Find square roots and multiply by coefficients.

$= (8 - 9)\sqrt{3}$ Combine like terms.

$= -\sqrt{3}$ Simplify.

b. $\sqrt[3]{16x} - \sqrt[3]{54x^4} = \sqrt[3]{8 \cdot 2x} - \sqrt[3]{27 \cdot x^3 \cdot 2x}$ Find cube factors.

$= 2\sqrt[3]{2x} - 3x\sqrt[3]{2x}$ Find cube roots.

$= (2 - 3x)\sqrt[3]{2x}$ Combine like terms.

CHECKPOINT Now try Exercise 71.

Rationalizing Denominators and Numerators

To rationalize a denominator or numerator of the form $a - b\sqrt{m}$ or $a + b\sqrt{m}$, multiply both numerator and denominator by a **conjugate:** $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are conjugates of each other. If $a = 0$, then the rationalizing factor for $\sqrt{m}$ is itself, $\sqrt{m}$. For cube roots, choose a rationalizing factor that generates a perfect cube.

Example 10 Rationalizing Single-Term Denominators

Rationalize the denominator of each expression.

a. $\frac{5}{2\sqrt{3}}$ **b.** $\frac{2}{\sqrt[3]{5}}$

Solution

a. $\frac{5}{2\sqrt{3}} = \frac{5}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$ $\sqrt{3}$ is rationalizing factor.

$= \frac{5\sqrt{3}}{2(3)}$ Multiply.

$= \frac{5\sqrt{3}}{6}$ Simplify.

b. $\frac{2}{\sqrt[3]{5}} = \frac{2}{\sqrt[3]{5}} \cdot \frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}$ $\sqrt[3]{5^2}$ is rationalizing factor.

$= \frac{2\sqrt[3]{5^2}}{\sqrt[3]{5^3}}$ Multiply.

$= \frac{2\sqrt[3]{25}}{5}$ Simplify.

CHECKPOINT Now try Exercise 79.

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Example 11 **Rationalizing a Denominator with Two Terms**

$$\frac{2}{3+\sqrt{7}} = \frac{2}{3+\sqrt{7}} \cdot \frac{3-\sqrt{7}}{3-\sqrt{7}}$$ Multiply numerator and denominator by conjugate of denominator.

$$= \frac{2(3-\sqrt{7})}{3(3)+3(-\sqrt{7})+\sqrt{7}(3)-(\sqrt{7})(\sqrt{7})}$$ Use Distributive Property.

$$= \frac{2(3-\sqrt{7})}{(3)^2-(\sqrt{7})^2}$$ Simplify.

$$= \frac{2(3-\sqrt{7})}{9-7}$$ Square terms of denominator.

$$= \frac{2(3-\sqrt{7})}{2} = 3-\sqrt{7}$$ Simplify.

CHECKPOINT Now try Exercise 81.

Sometimes it is necessary to rationalize the numerator of an expression. For instance, in Section P.5 you will use the technique shown in the next example to rationalize the numerator of an expression from calculus.

STUDY TIP

Do not confuse the expression $\sqrt{5}+\sqrt{7}$ with the expression $\sqrt{5+7}$. In general, $\sqrt{x+y}$ does not equal $\sqrt{x}+\sqrt{y}$. Similarly, $\sqrt{x^2+y^2}$ does not equal $x+y$.

Example 12 **Rationalizing a Numerator** ∫

$$\frac{\sqrt{5}-\sqrt{7}}{2} = \frac{\sqrt{5}-\sqrt{7}}{2} \cdot \frac{\sqrt{5}+\sqrt{7}}{\sqrt{5}+\sqrt{7}}$$ Multiply numerator and denominator by conjugate of numerator.

$$= \frac{(\sqrt{5})^2-(\sqrt{7})^2}{2(\sqrt{5}+\sqrt{7})}$$ Simplify.

$$= \frac{5-7}{2(\sqrt{5}+\sqrt{7})}$$ Square terms of numerator.

$$= \frac{-2}{2(\sqrt{5}+\sqrt{7})} = \frac{-1}{\sqrt{5}+\sqrt{7}}$$ Simplify.

CHECKPOINT Now try Exercise 85.

Rational Exponents

Definition of Rational Exponents

If a is a real number and n is a positive integer such that the principal nth root of a exists, then $a^{1/n}$ is defined as

$a^{1/n} = \sqrt[n]{a}$, where $1/n$ is the **rational exponent** of a.

Moreover, if m is a positive integer that has no common factor with n, then

$a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m$ and $a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$.

The symbol ∫ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

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STUDY TIP

Rational exponents can be tricky, and you must remember that the expression $b^{m/n}$ is not defined unless $\sqrt[n]{b}$ is a real number. This restriction produces some unusual-looking results. For instance, the number $(-8)^{1/3}$ is defined because $\sqrt[3]{-8} = -2$, but the number $(-8)^{2/6}$ is undefined because $\sqrt[6]{-8}$ is not a real number.

The numerator of a rational exponent denotes the *power* to which the base is raised, and the denominator denotes the *index* or the *root* to be taken.

Power
Index

$$b^{m/n} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$$

When you are working with rational exponents, the properties of integer exponents still apply. For instance,

$$2^{1/2}2^{1/3} = 2^{(1/2)+(1/3)} = 2^{5/6}.$$

Example 13 Changing from Radical to Exponential Form

a. $\sqrt{3} = 3^{1/2}$

b. $\sqrt{(3xy)^5} = \sqrt[2]{(3xy)^5} = (3xy)^{(5/2)}$

c. $2x\sqrt[4]{x^3} = (2x)(x^{3/4}) = 2x^{1+(3/4)} = 2x^{7/4}$

CHECKPOINT Now try Exercise 87.

Technology

There are four methods of evaluating radicals on most graphing calculators. For square roots, you can use the *square root key* $\boxed{\sqrt{\ }}$. For cube roots, you can use the *cube root key* $\boxed{\sqrt[3]{\ }}$. For other roots, you can first convert the radical to exponential form and then use the *exponential key* $\boxed{\wedge}$, or you can use the *xth root key* $\boxed{\sqrt[x]{\ }}$. Consult the user's guide for your calculator for specific keystrokes.

Example 14 Changing from Exponential to Radical Form

a. $(x^2 + y^2)^{3/2} = \left(\sqrt{x^2 + y^2}\right)^3 = \sqrt{(x^2 + y^2)^3}$

b. $2y^{3/4}z^{1/4} = 2(y^3z)^{1/4} = 2\sqrt[4]{y^3z}$

c. $a^{-3/2} = \dfrac{1}{a^{3/2}} = \dfrac{1}{\sqrt{a^3}}$

d. $x^{0.2} = x^{1/5} = \sqrt[5]{x}$

CHECKPOINT Now try Exercise 89.

Rational exponents are useful for evaluating roots of numbers on a calculator, for reducing the index of a radical, and for simplifying expressions in calculus.

Example 15 Simplifying with Rational Exponents

a. $(-32)^{-4/5} = \left(\sqrt[5]{-32}\right)^{-4} = (-2)^{-4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

b. $(-5x^{5/3})(3x^{-3/4}) = -15x^{(5/3)-(3/4)} = -15x^{11/12}, \quad x \neq 0$

c. $\sqrt[9]{a^3} = a^{3/9} = a^{1/3} = \sqrt[3]{a}$ Reduce index.

d. $\sqrt[3]{\sqrt{125}} = \sqrt[6]{125} = \sqrt[6]{(5)^3} = 5^{3/6} = 5^{1/2} = \sqrt{5}$

e. $(2x - 1)^{4/3}(2x - 1)^{-1/3} = (2x - 1)^{(4/3)-(1/3)}$

$= 2x - 1, \quad x \neq \dfrac{1}{2}$

f. $\dfrac{x - 1}{(x - 1)^{-1/2}} = \dfrac{x - 1}{(x - 1)^{-1/2}} \cdot \dfrac{(x - 1)^{1/2}}{(x - 1)^{1/2}}$

$= \dfrac{(x - 1)^{3/2}}{(x - 1)^0}$

$= (x - 1)^{3/2}, \quad x \neq 1$

CHECKPOINT Now try Exercise 99.

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P.2 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. In the exponential form a^n, n is the _____ and a is the _____.
2. A convenient way of writing very large or very small numbers is called _____ _____.
3. One of the two equal factors of a number is called a _____ _____ of the number.
4. The _____ _____ _____ of a number is the nth root that has the same sign as a, and is denoted by $\sqrt[n]{a}$.
5. In the radical form, $\sqrt[n]{a}$ the positive integer n is called the _____ of the radical and the number a is called the _____.
6. When an expression involving radicals has all possible factors removed, radical-free denominators, and a reduced index, it is in _____ _____.
7. The expressions $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are _____ of each other.
8. The process used to create a radical-free denominator is know as _____ the denominator.
9. In the expression $b^{m/n}$, m denotes the _____ to which the base is raised and n denotes the _____ or root to be taken.

In Exercises 1 and 2, write the expression as a repeated multiplication problem.

1. 8^5 **2.** $(-2)^7$

In Exercises 3 and 4, write the expression using exponential notation.

3. $(4.9)(4.9)(4.9)(4.9)(4.9)(4.9)$

4. $(-10)(-10)(-10)(-10)(-10)$

In Exercises 5–12, evaluate each expression.

5. (a) $3^2 \cdot 3$ (b) $3 \cdot 3^3$

6. (a) $\dfrac{5^5}{5^2}$ (b) $\dfrac{3^2}{3^4}$

7. (a) $(3^3)^0$ (b) -3^2

8. (a) $(2^3 \cdot 3^2)^2$ (b) $\left(-\frac{3}{5}\right)^3\left(\frac{5}{3}\right)^2$

9. (a) $\dfrac{3 \cdot 4^{-4}}{3^{-4} \cdot 4^{-1}}$ (b) $32(-2)^{-5}$

10. (a) $\dfrac{4 \cdot 3^{-2}}{2^{-2} \cdot 3^{-1}}$ (b) $(-2)^0$

11. (a) $2^{-1} + 3^{-1}$ (b) $(2^{-1})^{-2}$

12. (a) $3^{-1} + 2^{-2}$ (b) $(3^{-2})^2$

In Exercises 13–16, use a calculator to evaluate the expression. (If necessary, round your answer to three decimal places.)

13. $(-4)^3(5^2)$ **14.** $(8^{-4})(10^3)$

15. $\dfrac{3^6}{7^3}$ **16.** $\dfrac{4^3}{3^{-4}}$

In Exercises 17–24, evaluate the expression for the given value of x.

	Expression	*Value*
17.	$-3x^3$	$x = 2$
18.	$7x^{-2}$	$x = 4$
19.	$6x^0$	$x = 10$
20.	$5(-x)^3$	$x = 3$
21.	$2x^3$	$x = -3$
22.	$-3x^4$	$x = -2$
23.	$4x^2$	$x = -\frac{1}{2}$
24.	$5(-x)^3$	$x = -\frac{1}{3}$

In Exercises 25–30, simplify each expression.

25. (a) $(-5z)^3$ (b) $5x^4(x^2)$

26. (a) $(3x)^2$ (b) $(4x^3)^0$

27. (a) $6y^2(2y^0)^2$ (b) $\dfrac{3x^5}{x^3}$

28. (a) $(-z)^3(3z^4)$ (b) $\dfrac{25y^8}{10y^4}$

29. (a) $\dfrac{7x^2}{x^3}$ (b) $\dfrac{12(x+y)^3}{9(x+y)}$

30. (a) $\dfrac{r^4}{r^6}$ (b) $\left(\dfrac{4}{y}\right)^3\left(\dfrac{3}{y}\right)^4$

In Exercises 31–36, rewrite each expression with positive exponents and simplify.

31. (a) $(x+5)^0, \quad x \neq -5$ (b) $(2x^2)^{-2}$

32. (a) $(2x^5)^0, \quad x \neq 0$ (b) $(z+2)^{-3}(z+2)^{-1}$

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33. (a) $(-2x^2)^3(4x^3)^{-1}$ (b) $\left(\frac{x}{10}\right)^{-1}$

34. (a) $(4y^{-2})(8y^4)$ (b) $\left(\frac{x^{-3}y^4}{5}\right)^{-3}$

35. (a) $3^n \cdot 3^{2n}$ (b) $\left(\frac{a^{-2}}{b^{-2}}\right)\left(\frac{b}{a}\right)^3$

36. (a) $\frac{x^2 \cdot x^n}{x^3 \cdot x^n}$ (b) $\left(\frac{a^{-3}}{b^{-3}}\right)\left(\frac{a}{b}\right)^3$

In Exercises 37–40, write the number in scientific notation.

37. Land area of Earth: 57,300,000 square miles

38. Light year: 9,460,000,000,000 kilometers

39. Relative density of hydrogen: 0.0000899 gram per cubic centimeter

40. One micron (millionth of a meter): 0.00003937 inch

In Exercises 41–44, write the number in decimal notation.

41. Worldwide daily consumption of Coca-Cola: 4.568×10^9 ounces (Source: The Coca-Cola Company)

42. Interior temperature of the sun: 1.5×10^7 degrees Celsius

43. Charge of an electron: 1.6022×10^{-19} coulomb

44. Width of a human hair: 9.0×10^{-5} meter

In Exercises 45 and 46, evaluate each expression without using a calculator.

45. (a) $\sqrt{25 \times 10^8}$ (b) $\sqrt[3]{8 \times 10^{15}}$

46. (a) $(1.2 \times 10^7)(5 \times 10^{-3})$ (b) $\frac{(6.0 \times 10^8)}{(3.0 \times 10^{-3})}$

In Exercises 47–50, use a calculator to evaluate each expression. (Round your answer to three decimal places.)

47. (a) $750\left(1 + \frac{0.11}{365}\right)^{800}$

(b) $\frac{67{,}000{,}000 + 93{,}000{,}000}{0.0052}$

48. (a) $(9.3 \times 10^6)^3(6.1 \times 10^{-4})$

(b) $\frac{(2.414 \times 10^4)^6}{(1.68 \times 10^5)^5}$

49. (a) $\sqrt{4.5 \times 10^9}$ (b) $\sqrt[3]{6.3 \times 10^4}$

50. (a) $(2.65 \times 10^{-4})^{1/3}$ (b) $\sqrt{9 \times 10^{-4}}$

In Exercises 51–56, evaluate each expression without using a calculator.

51. (a) $\sqrt{9}$ (b) $\sqrt[3]{\frac{27}{8}}$

52. (a) $27^{1/3}$ (b) $36^{3/2}$

53. (a) $32^{-3/5}$ (b) $\left(\frac{16}{81}\right)^{-3/4}$

54. (a) $100^{-3/2}$ (b) $\left(\frac{9}{4}\right)^{-1/2}$

55. (a) $\left(-\frac{1}{64}\right)^{-1/3}$ (b) $\left(\frac{1}{\sqrt{32}}\right)^{-2/5}$

56. (a) $\left(-\frac{125}{27}\right)^{-1/3}$ (b) $-\left(\frac{1}{125}\right)^{-4/3}$

In Exercises 57–60, use a calculator to approximate the number. (Round your answer to three decimal places.)

57. (a) $\sqrt{57}$ (b) $\sqrt[5]{-27^3}$

58. (a) $\sqrt[3]{45^2}$ (b) $\sqrt[6]{125}$

59. (a) $(-12.4)^{-1.8}$ (b) $\left(5\sqrt{3}\right)^{-2.5}$

60. (a) $\frac{7 - (4.1)^{-3.2}}{2}$ (b) $\left(\frac{13}{3}\right)^{-3/2} - \left(-\frac{3}{2}\right)^{13/3}$

In Exercises 61 and 62, use the properties of radicals to simplify each expression.

61. (a) $\left(\sqrt[3]{4}\right)^3$ (b) $\sqrt[5]{96x^5}$

62. (a) $\sqrt{12} \cdot \sqrt{3}$ (b) $\sqrt[4]{(3x^2)^4}$

In Exercises 63–74, simplify each radical expression.

63. (a) $\sqrt{8}$ (b) $\sqrt[3]{54}$

64. (a) $\sqrt[3]{\frac{16}{27}}$ (b) $\sqrt{\frac{75}{4}}$

65. (a) $\sqrt{72x^3}$ (b) $\sqrt{\frac{18^2}{z^3}}$

66. (a) $\sqrt{54xy^4}$ (b) $\sqrt{\frac{32a^4}{b^2}}$

67. (a) $\sqrt[3]{16x^5}$ (b) $\sqrt{75x^2y^{-4}}$

68. (a) $\sqrt[4]{3x^4y^2}$ (b) $\sqrt[5]{160x^8z^4}$

69. (a) $2\sqrt{50} + 12\sqrt{8}$ (b) $10\sqrt{32} - 6\sqrt{18}$

70. (a) $4\sqrt{27} - \sqrt{75}$ (b) $\sqrt[3]{16} + 3\sqrt[3]{54}$

71. (a) $5\sqrt{x} - 3\sqrt{x}$ (b) $-2\sqrt{9y} + 10\sqrt{y}$

72. (a) $8\sqrt{49x} - 14\sqrt{100x}$ (b) $-3\sqrt{48x^2} + 7\sqrt{75x^2}$

73. (a) $3\sqrt{x+1} + 10\sqrt{x+1}$ (b) $7\sqrt{80x} - 2\sqrt{125x}$

74. (a) $-\sqrt{x^3 - 7} + 5\sqrt{x^3 - 7}$ (b) $11\sqrt{245x^3} - 9\sqrt{45x^3}$

In Exercises 75–78, complete the statement with <, =, or >.

75. $\sqrt{5} + \sqrt{3}$ ☐ $\sqrt{5+3}$ **76.** $\sqrt{\frac{3}{11}}$ ☐ $\frac{\sqrt{3}}{\sqrt{11}}$

77. 5 ☐ $\sqrt{3^2 + 2^2}$ **78.** 5 ☐ $\sqrt{3^2 + 4^2}$

In Exercises 79–82, rationalize the denominator of the expression. Then simplify your answer.

79. $\frac{1}{\sqrt{3}}$ **80.** $\frac{5}{\sqrt{10}}$

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81. $\dfrac{2}{5-\sqrt{3}}$ **82.** $\dfrac{3}{\sqrt{5}+\sqrt{6}}$

In Exercises 83–86, rationalize the numerator of the expression. Then simplify your answer.

83. $\dfrac{\sqrt{8}}{2}$ **84.** $\dfrac{\sqrt{2}}{3}$

85. $\dfrac{\sqrt{5}+\sqrt{3}}{3}$ **86.** $\dfrac{\sqrt{7}-3}{4}$

In Exercises 87–94, fill in the missing form of the expression.

	Radical Form	*Rational Exponent Form*
87.	$\sqrt{9}$	
88.	$\sqrt[3]{64}$	
89.		$32^{1/5}$
90.		$-(144^{1/2})$
91.	$\sqrt[3]{-216}$	
92.		$(-243)^{1/5}$
93.	$\left(\sqrt[4]{81}\right)^3$	
94.		$16^{5/4}$

In Exercises 95–98, perform the operations and simplify.

95. $\dfrac{(2x^2)^{3/2}}{2^{1/2}x^4}$ **96.** $\dfrac{x^{4/3}y^{2/3}}{(xy)^{1/3}}$

97. $\dfrac{x^{-3}\cdot x^{1/2}}{x^{3/2}\cdot x^{-1}}$ **98.** $\dfrac{5^{-1/2}\cdot 5x^{5/2}}{(5x)^{3/2}}$

In Exercises 99 and 100, reduce the index of each radical.

99. (a) $\sqrt[4]{3^2}$ (b) $\sqrt[6]{(x+1)^4}$

100. (a) $\sqrt[6]{x^3}$ (b) $\sqrt[4]{(3x^2)^4}$

In Exercises 101 and 102, write each expression as a single radical. Then simplify your answer.

101. (a) $\sqrt{\sqrt{32}}$ (b) $\sqrt{\sqrt[4]{2x}}$

102. (a) $\sqrt{\sqrt{243(x+1)}}$ (b) $\sqrt{\sqrt[3]{10a^7b}}$

103. ***Period of a Pendulum*** The period T (in seconds) of a pendulum is

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where L is the length of the pendulum (in feet). Find the period of a pendulum whose length is 2 feet.

104. ***Erosion*** A stream of water moving at the rate of v feet per second can carry particles of size $0.03\sqrt{v}$ inches. Find the size of the largest particle that can be carried by a stream flowing at the rate of $\frac{3}{4}$ foot per second.

Model It

105. ***Mathematical Modeling*** A funnel is filled with water to a height of h centimeters. The formula

$$t = 0.03[12^{5/2} - (12-h)^{5/2}], \quad 0 \le h \le 12$$

represents the amount of time t (in seconds) that it will take for the funnel to empty.

(a) Use the *table* feature of a graphing utility to find the times required for the funnel to empty for water heights of $h = 0$, $h = 1$, $h = 2$, . . . $h = 12$ centimeters.

(b) What value does t appear to be approaching as the height of the water becomes closer and closer to 12 centimeters?

106. ***Speed of Light*** The speed of light is approximately 11,180,000 miles per minute. The distance from the sun to Earth is approximately 93,000,000 miles. Find the time for light to travel from the sun to Earth.

Synthesis

True or False? **In Exercises 107 and 108, determine whether the statement is true or false. Justify your answer.**

107. $\dfrac{x^{k+1}}{x} = x^k$ **108.** $(a^n)^k = a^{n^k}$

109. Verify that $a^0 = 1$, $a \neq 0$. (*Hint:* Use the property of exponents $a^m/a^n = a^{m-n}$.)

110. Explain why each of the following pairs is not equal.

(a) $(3x)^{-1} \neq \dfrac{3}{x}$ (b) $y^3\cdot y^2 \neq y^6$

(c) $(a^2b^3)^4 \neq a^6b^7$ (d) $(a+b)^2 \neq a^2+b^2$

(e) $\sqrt{4x^2} \neq 2x$ (f) $\sqrt{2}+\sqrt{3} \neq \sqrt{5}$

111. ***Exploration*** List all possible digits that occur in the units place of the square of a positive integer. Use that list to determine whether $\sqrt{5233}$ is an integer.

112. ***Think About It*** Square the real number $2/\sqrt{5}$ and note that the radical is eliminated from the denominator. Is this equivalent to rationalizing the denominator? Why or why not?

The symbol indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

The symbol indicates an exercise or a part of an exercise in which you are instructed to use a graphing utility.

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P.3 Polynomials and Special Products

What you should learn

- Write polynomials in standard form.
- Add, subtract, and multiply polynomials.
- Use special products to multiply polynomials.
- Use polynomials to solve real-life problems.

Why you should learn it

Polynomials can be used to model and solve real-life problems. For instance, in Exercise 98 on page 30, polynomials are used to model the cost, revenue, and profit for producing and selling hats.



David Noton/Masterfile

Polynomials

The most common type of algebraic expression is the **polynomial.** Some examples are $2x + 5$, $3x^4 - 7x^2 + 2x + 4$, and $5x^2y^2 - xy + 3$. The first two are *polynomials in* x and the third is a *polynomial in* x *and* y. The terms of a polynomial in x have the form ax^k, where a is the **coefficient** and k is the **degree** of the term. For instance, the polynomial

$$2x^3 - 5x^2 + 1 = 2x^3 + (-5)x^2 + (0)x + 1$$

has coefficients 2, -5, 0, and 1.

Definition of a Polynomial in x

Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and let n be a nonnegative integer. A polynomial in x is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

where $a_n \neq 0$. The polynomial is of **degree** n, a_n is the **leading coefficient,** and a_0 is the **constant term.**

Polynomials with one, two, and three terms are called **monomials, binomials,** and **trinomials,** respectively. In **standard form,** a polynomial is written with descending powers of x.

Example 1 Writing Polynomials in Standard Form

	Polynomial	*Standard Form*	*Degree*
a.	$4x^2 - 5x^7 - 2 + 3x$	$-5x^7 + 4x^2 + 3x - 2$	7
b.	$4 - 9x^2$	$-9x^2 + 4$	2
c.	8	$8 \ (8 = 8x^0)$	0

CHECKPOINT Now try Exercise 11.

A polynomial that has all zero coefficients is called the zero polynomial, denoted by 0. No degree is assigned to this particular polynomial. For polynomials in more than one variable, the degree of a *term* is the sum of the exponents of the variables in the term. The degree of the *polynomial* is the highest degree of its terms. For instance, the degree of the polynomial $-2x^3y^6 + 4xy - x^7y^4$ is 11 because the sum of the exponents in the last term is the greatest. The leading coefficient of the polynomial is the coefficient of the highest-degree term. Expressions are not polynomials if a variable is underneath a radical or if a polynomial expression (with degree greater than 0) is in the denominator of a term. The following expressions are not polynomials.

$$x^3 - \sqrt{3x} = x^3 - (3x)^{1/2}$$ The exponent "1/2" is not an integer.

$$x^2 + \frac{5}{x} = x^2 + 5x^{-1}$$ The exponent "-1" is not a nonnegative integer.

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Operations with Polynomials

You can add and subtract polynomials in much the same way you add and subtract real numbers. Simply add or subtract the *like terms* (terms having the same variables to the same powers) by adding their coefficients. For instance, $-3xy^2$ and $5xy^2$ are like terms and their sum is

$$\begin{aligned} -3xy^2 + 5xy^2 &= (-3 + 5)xy^2 \\ &= 2xy^2. \end{aligned}$$

STUDY TIP

When an expression inside parentheses is preceded by a negative sign, remember to distribute the negative sign to each term inside the parentheses, as shown.

$$\begin{aligned} -(x^2 - x + 3) \\ = -x^2 + x - 3 \end{aligned}$$

Example 2 **Sums and Differences of Polynomials**

a. $(5x^3 - 7x^2 - 3) + (x^3 + 2x^2 - x + 8)$

$= (5x^3 + x^3) + (-7x^2 + 2x^2) - x + (-3 + 8)$ Group like terms.

$= 6x^3 - 5x^2 - x + 5$ Combine like terms.

b. $(7x^4 - x^2 - 4x + 2) - (3x^4 - 4x^2 + 3x)$

$= 7x^4 - x^2 - 4x + 2 - 3x^4 + 4x^2 - 3x$ Distributive Property

$= (7x^4 - 3x^4) + (-x^2 + 4x^2) + (-4x - 3x) + 2$ Group like terms.

$= 4x^4 + 3x^2 - 7x + 2$ Combine like terms.

CHECKPOINT Now try Exercise 33.

To find the *product* of two polynomials, use the left and right Distributive Properties. For example, if you treat $5x + 7$ as a single quantity, you can multiply $3x - 2$ by $5x + 7$ as follows.

$$\begin{aligned} (3x - 2)(5x + 7) &= 3x(5x + 7) - 2(5x + 7) \\ &= (3x)(5x) + (3x)(7) - (2)(5x) - (2)(7) \\ &= 15x^2 + 21x - 10x - 14 \end{aligned}$$

Product of **F**irst terms — Product of **O**uter terms — Product of **I**nner terms — Product of **L**ast terms

$$= 15x^2 + 11x - 14$$

Note in this **FOIL Method** (which can only be used to multiply two binomials) that the outer (O) and inner (I) terms are like terms and can be combined.

Example 3 **Finding a Product by the FOIL Method**

Use the FOIL Method to find the product of $2x - 4$ and $x + 5$.

Solution

$$\begin{aligned} (2x - 4)(x + 5) &= \overset{\text{F}}{2x^2} + \overset{\text{O}}{10x} - \overset{\text{I}}{4x} - \overset{\text{L}}{20} \\ &= 2x^2 + 6x - 20 \end{aligned}$$

CHECKPOINT Now try Exercise 55.

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When multiplying two polynomials, be sure to multiply each term of one polynomial by *each* term of the other. A vertical arrangement is helpful.

Example 4 **A Vertical Arrangement for Multiplication**

Multiply $x^2 - 2x + 2$ by $x^2 + 2x + 2$ using a vertical arrangement.

Solution

$$\begin{array}{rll} x^2 - 2x + 2 & & \text{Write in standard form.} \\ \times\ x^2 + 2x + 2 & & \text{Write in standard form.} \\ \hline x^4 - 2x^3 + 2x^2 & \Leftarrow & x^2(x^2 - 2x + 2) \\ 2x^3 - 4x^2 + 4x & \Leftarrow & 2x(x^2 - 2x + 2) \\ 2x^2 - 4x + 4 & \Leftarrow & 2(x^2 - 2x + 2) \\ \hline x^4 + 0x^3 + 0x^2 + 0x + 4 = x^4 + 4 & & \text{Combine like terms.} \end{array}$$

So, $(x^2 - 2x + 2)(x^2 + 2x + 2) = x^4 + 4$.

CHECKPOINT Now try Exercise 59.

Special Products

Some binomial products have special forms that occur frequently in algebra. You do not need to memorize these formulas because you can use the Distributive Property to multiply. However, becoming familiar with these formulas will enable you to manipulate the algebra more quickly.

Special Products

Let u and v be real numbers, variables, or algebraic expressions.

Special Product	*Example*
Sum and Difference of Same Terms	
$(u + v)(u - v) = u^2 - v^2$	$(x + 4)(x - 4) = x^2 - 4^2 = x^2 - 16$
Square of a Binomial	
$(u + v)^2 = u^2 + 2uv + v^2$	$(x + 3)^2 = x^2 + 2(x)(3) + 3^2 = x^2 + 6x + 9$
$(u - v)^2 = u^2 - 2uv + v^2$	$(3x - 2)^2 = (3x)^2 - 2(3x)(2) + 2^2 = 9x^2 - 12x + 4$
Cube of a Binomial	
$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$	$(x + 2)^3 = x^3 + 3x^2(2) + 3x(2^2) + 2^3 = x^3 + 6x^2 + 12x + 8$
$(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$	$(x - 1)^3 = x^3 - 3x^2(1) + 3x(1^2) - 1^3 = x^3 - 3x^2 + 3x - 1$

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Example 5 Sum and Difference of Same Terms

Find the product of $5x + 9$ and $5x - 9$.

Solution

The product of a sum and a difference of the *same* two terms has no middle term and takes the form $(u + v)(u - v) = u^2 - v^2$.

$$(5x + 9)(5x - 9) = (5x)^2 - 9^2 = 25x^2 - 81$$

CHECKPOINT Now try Exercise 61.

STUDY TIP

When squaring a binomial, note that the resulting middle term is always *twice* the product of the two terms.

Example 6 Square of a Binomial

Find $(6x - 5)^2$.

Solution

The square of a binomial has the form $(u - v)^2 = u^2 - 2uv + v^2$.

$$(6x - 5)^2 = (6x)^2 - 2(6x)(5) + 5^2 = 36x^2 - 60x + 25$$

CHECKPOINT Now try Exercise 67.

Example 7 Cube of a Binomial

Find $(3x + 2)^3$.

Solution

The cube of a binomial has the form

$$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3.$$

Note the *decrease* of powers of $u = 3x$ and the *increase* of powers of $v = 2$.

$$\begin{aligned}(3x + 2)^3 &= (3x)^3 + 3(3x)^2(2) + 3(3x)(2^2) + 2^3\\ &= 27x^3 + 54x^2 + 36x + 8\end{aligned}$$

CHECKPOINT Now try Exercise 69.

Example 8 The Product of Two Trinomials

Find the product of $x + y - 2$ and $x + y + 2$.

Solution

By grouping $x + y$ in parentheses, you can write the product of the trinomials as a special product.

Difference Sum

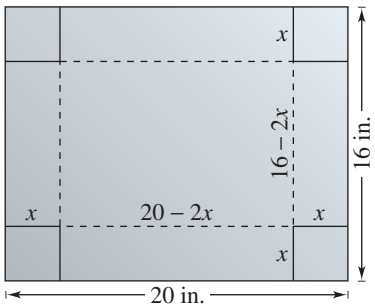
$$\begin{aligned}(x + y - 2)(x + y + 2) &= [(x + y) - 2][(x + y) + 2]\\ &= (x + y)^2 - 2^2 \qquad \text{Sum and difference of same terms}\\ &= x^2 + 2xy + y^2 - 4\end{aligned}$$

CHECKPOINT Now try Exercise 59.

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Application

Example 9 **Volume of a Box**



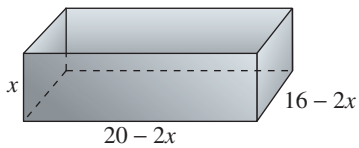


FIGURE P.8

An open box is made by cutting squares from the corners of a piece of metal that is 16 inches by 20 inches, as shown in Figure P.8. The edge of each cut-out square is x inches. Find the volume of the box when $x = 1$, $x = 2$, and $x = 3$.

Solution

The volume of a rectangular box is equal to the product of its length, width, and height. From the figure, the length is $20 - 2x$, the width is $16 - 2x$, and the height is x. So, the volume of the box is

$$\begin{aligned} \text{Volume} &= (20 - 2x)(16 - 2x)(x) \\ &= (320 - 72x + 4x^2)(x) \\ &= 320x - 72x^2 + 4x^3. \end{aligned}$$

When $x = 1$ inch, the volume of the box is

$$\begin{aligned} \text{Volume} &= 320(1) - 72(1)^2 + 4(1)^3 \\ &= 252 \text{ cubic inches.} \end{aligned}$$

When $x = 2$ inches, the volume of the box is

$$\begin{aligned} \text{Volume} &= 320(2) - 72(2)^2 + 4(2)^3 \\ &= 384 \text{ cubic inches.} \end{aligned}$$

When $x = 3$ inches, the volume of the box is

$$\begin{aligned} \text{Volume} &= 320(3) - 72(3)^2 + 4(3)^3 \\ &= 420 \text{ cubic inches.} \end{aligned}$$

CHECKPOINT Now try Exercise 101.

WRITING ABOUT MATHEMATICS

Mathematical Experiment In Example 9, the volume of the open box is given by

$$\text{Volume} = 320x - 72x^2 + 4x^3.$$

You want to create a box that has as much volume as possible. From Example 9, you know that by cutting one-, two-, and three-inch squares from the corners, you can create boxes whose volumes are 252, 384, and 420 cubic inches, respectively. What are the possible values of x that make sense in the problem? Write your answer as an interval. Try several other values of x to find the size of the squares that should be cut from the corners to produce a box that has maximum volume. Write a summary of your findings.

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P.3 Exercises

VOCABULARY CHECK

In Exercises 1–5, fill in the blanks.

1. For the polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, the degree is _____, the leading coefficient is _____, and the constant term is _____.
2. A polynomial in x in standard form is written with _____ powers of x.
3. A polynomial with one term is called a _____, while a polynomial with two terms is called a _____, and a polynomial with three terms is called a _____.
4. To add or subtract polynomials, add or subtract the _____ _____ by adding their coefficients.
5. The letters in "FOIL" stand for the following.

   F _____ O _____ I _____ L _____

In Exercises 6–8, match the special product form with its name.

6. $(u + v)(u - v) = u^2 - v^2$
7. $(u + v)^2 = u^2 + 2uv + v^2$
8. $(u - v)^2 = u^2 - 2uv + v^2$

(a) A binomial sum squared
(b) A binomial difference squared
(c) The sum and difference of same terms

In Exercises 1–6, match the polynomial with its description. [The polynomials are labeled (a), (b), (c), (d), (e), and (f).]

(a) $3x^2$ (b) $1 - 2x^3$
(c) $x^3 + 3x^2 + 3x + 1$ (d) 12
(e) $-3x^5 + 2x^3 + x$ (f) $\frac{2}{3}x^4 + x^2 + 10$

1. A polynomial of degree 0
2. A trinomial of degree 5
3. A binomial with leading coefficient -2
4. A monomial of positive degree
5. A trinomial with leading coefficient $\frac{2}{3}$
6. A third-degree polynomial with leading coefficient 1

In Exercises 7–10, write a polynomial that fits the description. (There are many correct answers.)

7. A third-degree polynomial with leading coefficient -2
8. A fifth-degree polynomial with leading coefficient 6
9. A fourth-degree binomial with a negative leading coefficient
10. A third-degree binomial with an even leading coefficient

In Exercises 11–22, (a) write the polynomial in standard form, (b) identify the degree and leading coefficient of the polynomial, and (c) state whether the polynomial is a monomial, a binomial, or a trinomial.

11. $14x - \frac{1}{2}x^5$
12. $2x^2 - x + 1$
13. $-3x^4 + 2x^2 - 5$
14. $7x$
15. $x^5 - 1$
16. $-y + 25y^2 + 1$
17. 3
18. $t^2 + 9$
19. $1 + 6x^4 - 4x^5$
20. $3 + 2x$
21. $4x^3y$
22. $-x^5y + 2x^2y^2 + xy^4$

In Exercises 23–28, determine whether the expression is a polynomial. If so, write the polynomial in standard form.

23. $2x - 3x^3 + 8$
24. $2x^3 + x - 3x^{-1}$
25. $\dfrac{3x + 4}{x}$
26. $\dfrac{x^2 + 2x - 3}{2}$
27. $y^2 - y^4 + y^3$
28. $\sqrt{y^2 - y^4}$

In Exercises 29–46, perform the operation and write the result in standard form.

29. $(6x + 5) - (8x + 15)$
30. $(2x^2 + 1) - (x^2 - 2x + 1)$
31. $-(x^3 - 2) + (4x^3 - 2x)$
32. $-(5x^2 - 1) - (-3x^2 + 5)$
33. $(15x^2 - 6) - (-8.3x^3 - 14.7x^2 - 17)$
34. $(15.2x^4 - 18x - 19.1) - (13.9x^4 - 9.6x + 15)$
35. $5z - [3z - (10z + 8)]$
36. $(y^3 + 1) - [(y^2 + 1) + (3y - 7)]$
37. $3x(x^2 - 2x + 1)$
38. $y^2(4y^2 + 2y - 3)$
39. $-5z(3z - 1)$
40. $(-3x)(5x + 2)$
41. $(1 - x^3)(4x)$
42. $-4x(3 - x^3)$
43. $(2.5x^2 + 3)(3x)$
44. $(2 - 3.5y)(2y^3)$
45. $-4x\left(\frac{1}{8}x + 3\right)$
46. $2y\left(4 - \frac{7}{8}y\right)$

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In Exercises 47–54, perform the operation.

47. Add $7x^3 - 2x^2 + 8$ and $-3x^3 - 4$.

48. Add $2x^5 - 3x^3 + 2x + 3$ and $4x^3 + x - 6$.

49. Subtract $x - 3$ from $5x^2 - 3x + 8$.

50. Subtract $-t^4 + 0.5t^2 - 5.6$ from $0.6t^4 - 2t^2$.

51. Multiply $-6x^2 + 15x - 4$ and $5x + 3$.

52. Multiply $4x^4 + x^3 - 6x^2 + 9$ and $x^2 + 2x + 3$.

53. $(x^2 + 9)(x^2 - x - 4)$

54. $(x - 2)(x^2 + 2x + 4)$

In Exercises 55–92, multiply or find the special product.

55. $(x + 3)(x + 4)$

56. $(x - 5)(x + 10)$

57. $(3x - 5)(2x + 1)$

58. $(7x - 2)(4x - 3)$

59. $(x^2 - x + 1)(x^2 + x + 1)$

60. $(x^2 + 3x - 2)(x^2 - 3x - 2)$

61. $(x + 10)(x - 10)$

62. $(2x + 3)(2x - 3)$

63. $(x + 2y)(x - 2y)$

64. $(2x + 3y)(2x - 3y)$

65. $(2x + 3)^2$

66. $(4x + 5)^2$

67. $(2x - 5y)^2$

68. $(5 - 8x)^2$

69. $(x + 1)^3$

70. $(x - 2)^3$

71. $(2x - y)^3$

72. $(3x + 2y)^3$

73. $(4x^3 - 3)^2$

74. $(8x + 3)^2$

75. $[(m - 3) + n][(m - 3) - n]$

76. $[(x + y) + 1][(x + y) - 1]$

77. $[(x - 3) + y]^2$

78. $[(x + 1) - y]^2$

79. $(2r^2 - 5)(2r^2 + 5)$

80. $(3a^3 - 4b^2)(3a^3 + 4b^2)$

81. $\left(\frac{1}{2}x - 3\right)^2$

82. $\left(\frac{2}{3}t + 5\right)^2$

83. $\left(\frac{1}{3}x - 2\right)\left(\frac{1}{3}x + 2\right)$

84. $\left(2x + \frac{1}{5}\right)\left(2x - \frac{1}{5}\right)$

85. $(1.2x + 3)^2$

86. $(1.5y - 3)^2$

87. $(1.5x - 4)(1.5x + 4)$

88. $(2.5y + 3)(2.5y - 3)$

89. $5x(x + 1) - 3x(x + 1)$

90. $(2x - 1)(x + 3) + 3(x + 3)$

91. $(u + 2)(u - 2)(u^2 + 4)$

92. $(x + y)(x - y)(x^2 + y^2)$

In Exercises 93–96, find the product. (The expressions are not polynomials, but the formulas can still be used.)

93. $\left(\sqrt{x} + \sqrt{y}\right)\left(\sqrt{x} - \sqrt{y}\right)$

94. $\left(5 + \sqrt{x}\right)\left(5 - \sqrt{x}\right)$

95. $\left(x - \sqrt{5}\right)^2$

96. $\left(x + \sqrt{3}\right)^2$

97. ***Cost, Revenue, and Profit*** An electronics manufacturer can produce and sell x radios per week. The total cost C (in dollars) for producing x radios is $C = 73x + 25{,}000$ and the total revenue R (in dollars) is $R = 95x$.

(a) Find the profit P in terms of x.

(b) Find the profit obtained by selling 5000 radios per week.

98. ***Cost, Revenue, and Profit*** An artisan can produce and sell x hats per month. The total cost C (in dollars) for producing x hats is $C = 460 + 12x$ and the total revenue R (in dollars) is $R = 36x$.

(a) Find the profit P in terms of x.

(b) Find the profit obtained by selling 42 hats per month.

99. ***Compound Interest*** After 2 years, an investment of \$500 compounded annually at an interest rate r will yield an amount of $500(1 + r)^2$.

(a) Write this polynomial in standard form.

(b) Use a calculator to evaluate the polynomial for the values of r shown in the table.

r	$2\frac{1}{2}\%$	3%	4%	$4\frac{1}{2}\%$	5%
$500(1 + r)^2$					

(c) What conclusion can you make from the table?

100. ***Compound Interest*** After 3 years, an investment of \$1200 compounded annually at an interest rate r will yield an amount of $1200(1 + r)^3$.

(a) Write this polynomial in standard form.

(b) Use a calculator to evaluate the polynomial for the values of r shown in the table.

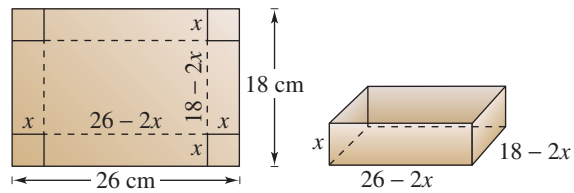
r	2%	3%	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$
$1200(1 + r)^3$					

(c) What conclusion can you make from the table?

101. ***Volume of a Box*** A take-out fast-food restaurant is constructing an open box by cutting squares from the corners of a piece of cardboard that is 18 centimeters by 26 centimeters (see figure). The edge of each cut-out square is x centimeters.

(a) Find the volume of the box in terms of x.

(b) Find the volume when $x = 1$, $x = 2$, and $x = 3$.



102. ***Volume of a Box*** An overnight shipping company is designing a closed box by cutting along the solid lines and folding along the broken lines on the rectangular piece of corrugated cardboard shown in the figure. The length and width of the rectangle are 45 centimeters and 15 centimeters, respectively.

(a) Find the volume of the shipping box in terms of x.

(b) Find the volume when $x = 3$, $x = 5$, and $x = 7$.

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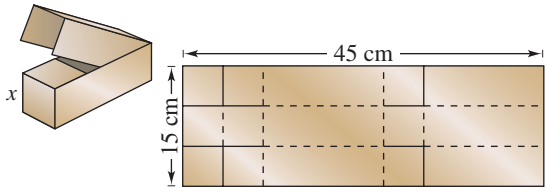
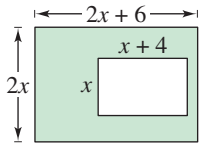


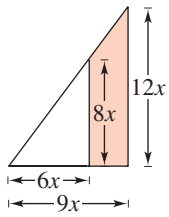
FIGURE FOR 102

103. ***Geometry*** Find the area of the shaded region in each figure. Write your result as a polynomial in standard form.

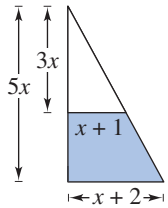
(a)



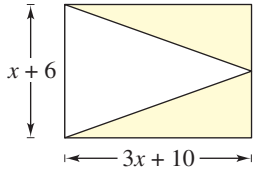
(b)



(c)

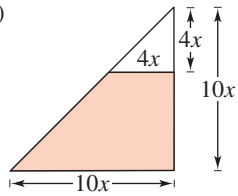


(d)

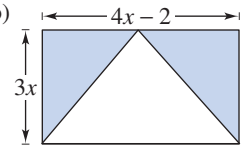


104. ***Geometry*** Find the area of the shaded region in each figure. Write your result as a polynomial in standard form.

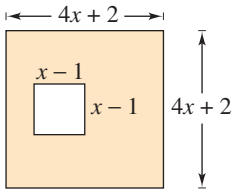
(a)



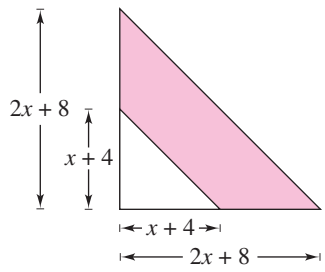
(b)



(c)

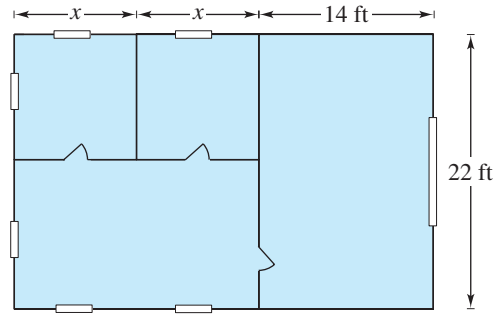


(d)

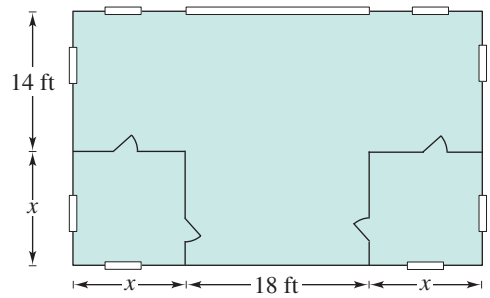


Geometry **In Exercises 105 and 106, find a polynomial that represents the total number of square feet for the floor plan shown in the figure.**

105.



106.



107. ***Engineering*** A uniformly distributed load is placed on a one-inch-wide steel beam. When the span of the beam is x feet and its depth is 6 inches, the safe load S (in pounds) is approximated by

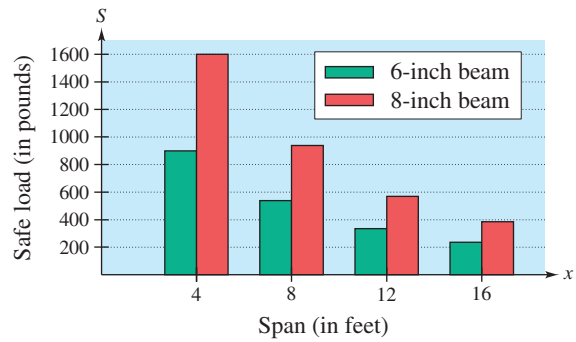
$$S_6 = (0.06x^2 - 2.42x + 38.71)^2.$$

When the depth is 8 inches, the safe load is approximated by

$$S_8 = (0.08x^2 - 3.30x + 51.93)^2.$$

(a) Use the bar graph to estimate the difference in the safe loads for these two beams when the span is 12 feet.

(b) How does the difference in safe load change as the span increases?



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Model It

108. ***Stopping Distance*** The stopping distance of an automobile is the distance traveled during the driver's reaction time plus the distance traveled after the brakes are applied. In an experiment, these distances were measured (in feet) when the automobile was traveling at a speed of x miles per hour on dry, level pavement, as shown in the bar graph. The distance traveled during the reaction time R was

$R = 1.1x$

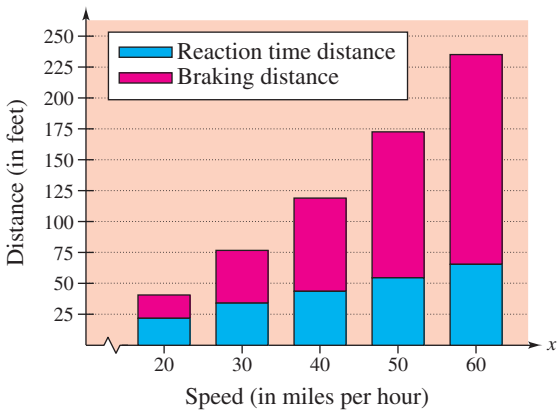
and the braking distance B was

$B = 0.0475x^2 - 0.001x + 0.23.$

(a) Determine the polynomial that represents the total stopping distance T.

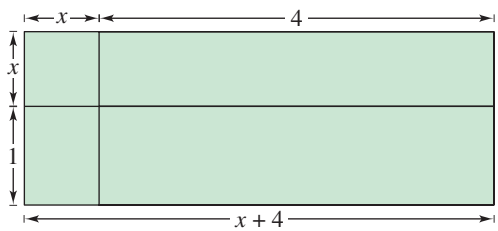
(b) Use the result of part (a) to estimate the total stopping distance when $x = 30$, $x = 40$, and $x = 55$ miles per hour.

(c) Use the bar graph to make a statement about the total stopping distance required for increasing speeds.

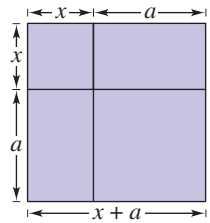


Geometry **In Exercises 109 and 110, use the area model to write two different expressions for the area. Then equate the two expressions and name the algebraic property that is illustrated.**

109.



110.



Synthesis

True or False? **In Exercises 111 and 112, determine whether the statement is true or false. Justify your answer.**

111. The product of two binomials is always a second-degree polynomial.

112. The sum of two binomials is always a binomial.

113. Find the degree of the product of two polynomials of degrees m and n.

114. Find the degree of the sum of two polynomials of degrees m and n if $m < n$.

115. ***Writing*** A student's homework paper included the following.

$(x - 3)^2 = x^2 + 9$

Write a paragraph fully explaining the error and give the correct method for squaring a binomial.

116. A third-degree polynomial and a fourth-degree polynomial are added.

(a) Can the sum be a fourth-degree polynomial? Explain or give an example.

(b) Can the sum be a second-degree polynomial? Explain or give an example.

(c) Can the sum be a seventh-degree polynomial? Explain or give an example.

117. ***Think About It*** Must the sum of two second-degree polynomials be a second-degree polynomial? If not, give an example.

118. ***Think About It*** When the polynomial $-x^3 + 3x^2 + 2x - 1$ is subtracted from an unknown polynomial, the difference is $5x^2 + 8$. If it is possible, find the unknown polynomial.

119. ***Logical Reasoning*** Verify that $(x + y)^2$ is not equal to $x^2 + y^2$ by letting $x = 3$ and $y = 4$ and evaluating both expressions. Are there any values of x and y for which $(x + y)^2 = x^2 + y^2$? Explain.

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P.4 Factoring Polynomials

What you should learn

- Remove common factors from polynomials.
- Factor special polynomial forms.
- Factor trinomials as the product of two binomials.
- Factor polynomials by grouping.

Why you should learn it

Polynomial factoring can be used to solve real-life problems. For instance, in Exercise 135 on page 40, factoring is used to develop an alternative model for the rate of change of an autocatalytic chemical reaction.



Mitch Wejnarowicz/The Image Works

Polynomials with Common Factors

The process of writing a polynomial as a product is called **factoring.** It is an important tool for solving equations and for simplifying rational expressions.

Unless noted otherwise, when you are asked to factor a polynomial, you can assume that you are looking for factors with integer coefficients. If a polynomial cannot be factored using integer coefficients, then it is **prime** or **irreducible over the integers.** For instance, the polynomial $x^2 - 3$ is irreducible over the integers. Over the *real numbers*, this polynomial can be factored as

$$x^2 - 3 = \left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right).$$

A polynomial is **completely factored** when each of its factors is prime. For instance

$$x^3 - x^2 + 4x - 4 = (x - 1)(x^2 + 4)$$ Completely factored

is completely factored, but

$$x^3 - x^2 - 4x + 4 = (x - 1)(x^2 - 4)$$ Not completely factored

is not completely factored. Its complete factorization is

$$x^3 - x^2 - 4x + 4 = (x - 1)(x + 2)(x - 2).$$

The simplest type of factoring involves a polynomial that can be written as the product of a monomial and another polynomial. The technique used here is the Distributive Property, $a(b + c) = ab + ac$, in the *reverse* direction.

$$ab + ac = a(b + c)$$ a is a common factor.

Removing (factoring out) any common factors is the first step in completely factoring a polynomial.

Example 1 Removing Common Factors

Factor each expression.

a. $6x^3 - 4x$

b. $-4x^2 + 12x - 16$

c. $(x - 2)(2x) + (x - 2)(3)$

Solution

a. $6x^3 - 4x = 2x(3x^2) - 2x(2)$ $2x$ is a common factor.

$= 2x(3x^2 - 2)$

b. $-4x^2 + 12x - 16 = -4(x^2) + (-4)(-3x) + (-4)4$ -4 is a common factor.

$= -4(x^2 - 3x + 4)$

c. $(x - 2)(2x) + (x - 2)(3) = (x - 2)(2x + 3)$ $(x - 2)$ is a common factor.

CHECKPOINT Now try Exercise 7.

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Factoring Special Polynomial Forms

Some polynomials have special forms that arise from the special product forms on page 26. You should learn to recognize these forms so that you can factor such polynomials easily.

Factoring Special Polynomial Forms

Factored Form	*Example*
Difference of Two Squares	
$u^2 - v^2 = (u + v)(u - v)$	$9x^2 - 4 = (3x)^2 - 2^2 = (3x + 2)(3x - 2)$
Perfect Square Trinomial	
$u^2 + 2uv + v^2 = (u + v)^2$	$x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2 = (x + 3)^2$
$u^2 - 2uv + v^2 = (u - v)^2$	$x^2 - 6x + 9 = x^2 - 2(x)(3) + 3^2 = (x - 3)^2$
Sum or Difference of Two Cubes	
$u^3 + v^3 = (u + v)(u^2 - uv + v^2)$	$x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - 2x + 4)$
$u^3 - v^3 = (u - v)(u^2 + uv + v^2)$	$27x^3 - 1 = (3x)^3 - 1^3 = (3x - 1)(9x^2 + 3x + 1)$

One of the easiest special polynomial forms to factor is the difference of two squares. The factored form is always a set of *conjugate pairs*.

$$u^2 - v^2 = (u + v)(u - v)$$

Difference — Opposite signs — Conjugate pairs

To recognize perfect square terms, look for coefficients that are squares of integers and variables raised to *even powers*.

STUDY TIP

In Example 2, note that the first step in factoring a polynomial is to check for any common factors. Once the common factors are removed, it is often possible to recognize patterns that were not immediately obvious.

Example 2 **Removing a Common Factor First**

$$\begin{aligned} 3 - 12x^2 &= 3(1 - 4x^2) && \text{3 is a common factor.} \\ &= 3[1^2 - (2x)^2] \\ &= 3(1 + 2x)(1 - 2x) && \text{Difference of two squares} \end{aligned}$$

CHECKPOINT Now try Exercise 21.

Example 3 **Factoring the Difference of Two Squares**

a. $(x + 2)^2 - y^2 = [(x + 2) + y][(x + 2) - y]$
$= (x + 2 + y)(x + 2 - y)$

b.
$$\begin{aligned} 16x^4 - 81 &= (4x^2)^2 - 9^2 \\ &= (4x^2 + 9)(4x^2 - 9) && \text{Difference of two squares} \\ &= (4x^2 + 9)[(2x)^2 - 3^2] \\ &= (4x^2 + 9)(2x + 3)(2x - 3) && \text{Difference of two squares} \end{aligned}$$

CHECKPOINT Now try Exercise 25.

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A **perfect square trinomial** is the square of a binomial, and it has the following form.

$$u^2 + 2uv + v^2 = (u + v)^2 \quad \text{or} \quad u^2 - 2uv + v^2 = (u - v)^2$$

Like signs Like signs

Note that the first and last terms are squares and the middle term is twice the product of u and v.

Example 4 Factoring Perfect Square Trinomials

Factor each trinomial.

a. $x^2 - 10x + 25$

b. $16x^2 + 24x + 9$

Solution

a. $x^2 - 10x + 25 = x^2 - 2(x)(5) + 5^2 = (x - 5)^2$

b. $16x^2 + 24x + 9 = (4x)^2 + 2(4x)(3) + 3^2 = (4x + 3)^2$

CHECKPOINT Now try Exercise 31.

Exploration

Rewrite $u^6 - v^6$ as the difference of two squares. Then find a formula for completely factoring $u^6 - v^6$. Use your formula to factor $x^6 - 1$ and $x^6 - 64$ completely.

The next two formulas show the sums and differences of cubes. Pay special attention to the signs of the terms.

Like signs Like signs

$$u^3 + v^3 = (u + v)(u^2 - uv + v^2) \qquad u^3 - v^3 = (u - v)(u^2 + uv + v^2)$$

Unlike signs Unlike signs

Example 5 Factoring the Difference of Cubes

Factor $x^3 - 27$.

Solution

$$x^3 - 27 = x^3 - 3^3 \qquad \text{Rewrite 27 as } 3^3.$$
$$= (x - 3)(x^2 + 3x + 9) \qquad \text{Factor.}$$

CHECKPOINT Now try Exercise 39.

Example 6 Factoring the Sum of Cubes

a. $y^3 + 8 = y^3 + 2^3 \qquad$ Rewrite 8 as 2^3.

$= (y + 2)(y^2 - 2y + 4) \qquad$ Factor.

b. $3(x^3 + 64) = 3(x^3 + 4^3) \qquad$ Rewrite 64 as 4^3.

$= 3(x + 4)(x^2 - 4x + 16) \qquad$ Factor.

CHECKPOINT Now try Exercise 41.

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Trinomials with Binomial Factors

To factor a trinomial of the form $ax^2 + bx + c$, use the following pattern.

Factors of a

$$ax^2 + bx + c = (\square x + \square)(\square x + \square)$$

Factors of c

The goal is to find a combination of factors of a and c such that the outer and inner products add up to the middle term bx. For instance, in the trinomial $6x^2 + 17x + 5$, you can write all possible factorizations and determine which one has outer and inner products that add up to $17x$.

$$(6x + 5)(x + 1),\ (6x + 1)(x + 5),\ (2x + 1)(3x + 5),\ (2x + 5)(3x + 1)$$

You can see that $(2x + 5)(3x + 1)$ is the correct factorization because the outer (O) and inner (I) products add up to $17x$.

F O I L O + I

$$(2x + 5)(3x + 1) = 6x^2 + 2x + 15x + 5 = 6x^2 + 17x + 5.$$

Example 7 Factoring a Trinomial: Leading Coefficient Is 1

Factor $x^2 - 7x + 12$.

Solution

The possible factorizations are

$$(x - 2)(x - 6), \quad (x - 1)(x - 12), \quad \text{and} \quad (x - 3)(x - 4).$$

Testing the middle term, you will find the correct factorization to be

$$x^2 - 7x + 12 = (x - 3)(x - 4).$$

CHECKPOINT Now try Exercise 51.

Example 8 Factoring a Trinomial: Leading Coefficient Is Not 1

Factor $2x^2 + x - 15$.

Solution

The eight possible factorizations are as follows.

$$(2x - 1)(x + 15) \qquad (2x + 1)(x - 15)$$
$$(2x - 3)(x + 5) \qquad (2x + 3)(x - 5)$$
$$(2x - 5)(x + 3) \qquad (2x + 5)(x - 3)$$
$$(2x - 15)(x + 1) \qquad (2x + 15)(x - 1)$$

Testing the middle term, you will find the correct factorization to be

$$2x^2 + x - 15 = (2x - 5)(x + 3). \qquad \text{O} + \text{I} = 6x - 5x = x$$

CHECKPOINT Now try Exercise 59.

STUDY TIP

Factoring a trinomial can involve trial and error. However, once you have produced the factored form, it is an easy matter to check your answer. For instance, you can verify the factorization in Example 7 by multiplying out the expression $(x - 3)(x - 4)$ to see that you obtain the original trinomial, $x^2 - 7x + 12$.

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Factoring by Grouping

Sometimes polynomials with more than three terms can be factored by a method called **factoring by grouping.** It is not always obvious which terms to group, and sometimes several different groupings will work.

Example 9 Factoring by Grouping

Use factoring by grouping to factor $x^3 - 2x^2 - 3x + 6$.

Solution

$$\begin{aligned} x^3 - 2x^2 - 3x + 6 &= (x^3 - 2x^2) - (3x - 6) && \text{Group terms.} \\ &= x^2(x - 2) - 3(x - 2) && \text{Factor each group.} \\ &= (x - 2)(x^2 - 3) && \text{Distributive Property} \end{aligned}$$

CHECKPOINT Now try Exercise 67.

STUDY TIP

Another way to factor the polynomial in Example 9 is to group the terms as follows.

$$\begin{aligned} &x^3 - 2x^2 - 3x + 6 \\ &= (x^3 - 3x) - (2x^2 - 6) \\ &= x(x^2 - 3) - 2(x^2 - 3) \\ &= (x^2 - 3)(x - 2) \end{aligned}$$

As you can see, you obtain the same result as in Example 9.

Factoring a trinomial can involve quite a bit of trial and error. Some of this trial and error can be lessened by using factoring by grouping. The key to this method of factoring is knowing how to rewrite the middle term. In general, to factor a trinomial $ax^2 + bx + c$ by grouping, choose factors of the product ac that add up to b and use these factors to rewrite the middle term. This technique is illustrated in Example 10.

Example 10 Factoring a Trinomial by Grouping

Use factoring by grouping to factor $2x^2 + 5x - 3$.

Solution

In the trinomial $2x^2 + 5x - 3$, $a = 2$ and $c = -3$, which implies that the product ac is -6. Now, -6 factors as $(6)(-1)$ and $6 - 1 = 5 = b$. So, you can rewrite the middle term as $5x = 6x - x$. This produces the following.

$$\begin{aligned} 2x^2 + 5x - 3 &= 2x^2 + 6x - x - 3 && \text{Rewrite middle term.} \\ &= (2x^2 + 6x) - (x + 3) && \text{Group terms.} \\ &= 2x(x + 3) - (x + 3) && \text{Factor groups.} \\ &= (x + 3)(2x - 1) && \text{Distributive Property} \end{aligned}$$

So, the trinomial factors as $2x^2 + 5x - 3 = (x + 3)(2x - 1)$.

CHECKPOINT Now try Exercise 73.

Guidelines for Factoring Polynomials

1. Factor out any common factors using the Distributive Property.
2. Factor according to one of the special polynomial forms.
3. Factor as $ax^2 + bx + c = (mx + r)(nx + s)$.
4. Factor by grouping.

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P.4 Exercises

VOCABULARY CHECK

In Exercises 1–4, fill in the blanks.

1. The process of writing a polynomial as a product is called _____.
2. If a polynomial cannot be factored with integer coefficients it is called prime or _____ over the integers.
3. A polynomial is _____ _____ when each of its factors is prime.
4. If a polynomial has more than three terms, a method of factoring called _____ _____ _____ may be used.
5. Match the factored form of the polynomial with its name.

(a) $u^2 - v^2 = (u + v)(u - v)$ (i) Perfect square trinomial

(b) $u^3 - v^3 = (u - v)(u^2 + uv + v^2)$ (ii) Difference of two squares

(c) $u^2 - 2uv + v^2 = (u - v)^2$ (iii) Difference of two cubes

---

In Exercises 1–4, find the greatest common factor of the expressions.

1. 90, 300
2. 36, 84, 294
3. $12x^2y^3, 18x^2y, 24x^3y^2$
4. $15(x + 2)^3, 42x(x + 2)^2$

In Exercises 5–12, factor out the common factor.

5. $3x + 6$
6. $5y - 30$
7. $2x^3 - 6x$
8. $4x^3 - 6x^2 + 12x$
9. $x(x - 1) + 6(x - 1)$
10. $3x(x + 2) - 4(x + 2)$
11. $(x + 3)^2 - 4(x + 3)$
12. $(3x - 1)^2 + (3x - 1)$

In Exercises 13–18, find the greatest common factor such that the remaining factors have only integer coefficients.

13. $\frac{1}{2}x + 4$
14. $\frac{1}{3}y + 5$
15. $\frac{1}{2}x^3 + 2x^2 - 5x$
16. $\frac{1}{3}y^4 - 5y^2 + 2y$
17. $\frac{2}{3}x(x - 3) - 4(x - 3)$
18. $\frac{4}{5}y(y + 1) - 2(y + 1)$

In Exercises 19–28, completely factor the difference of two squares.

19. $x^2 - 81$
20. $x^2 - 49$
21. $32y^2 - 18$
22. $4 - 36y^2$
23. $16x^2 - \frac{1}{9}$
24. $\frac{4}{25}y^2 - 64$
25. $(x - 1)^2 - 4$
26. $25 - (z + 5)^2$
27. $9u^2 - 4v^2$
28. $25x^2 - 16y^2$

In Exercises 29–38, factor the perfect square trinomial.

29. $x^2 - 4x + 4$
30. $x^2 + 10x + 25$
31. $4t^2 + 4t + 1$
32. $9x^2 - 12x + 4$
33. $25y^2 - 10y + 1$
34. $36y^2 - 108y + 81$
35. $9u^2 + 24uv + 16v^2$
36. $4x^2 - 4xy + y^2$
37. $x^2 - \frac{4}{3}x + \frac{4}{9}$
38. $z^2 + z + \frac{1}{4}$

In Exercises 39–46, factor the sum or difference of cubes.

39. $x^3 - 8$
40. $x^3 - 27$
41. $y^3 + 64$
42. $z^3 + 125$
43. $8t^3 - 1$
44. $27x^3 + 8$
45. $u^3 + 27v^3$
46. $64x^3 - y^3$

In Exercises 47–50, factor a negative real number from the polynomial and then write the polynomial factor in standard form.

47. $25 - 5x^2$
48. $5 + 3y^2 - y^3$
49. $-2t^3 + 4t + 6$
50. $-3x^5 - 3x^2 + 6x + 9$

In Exercises 51–64, factor the trinomial.

51. $x^2 + x - 2$
52. $x^2 + 5x + 6$
53. $s^2 - 5s + 6$
54. $t^2 - t - 6$
55. $20 - y - y^2$
56. $24 + 5z - z^2$
57. $x^2 - 30x + 200$
58. $x^2 - 13x + 42$
59. $3x^2 - 5x + 2$
60. $2x^2 - x - 1$
61. $5x^2 + 26x + 5$
62. $12x^2 + 7x + 1$
63. $-9z^2 + 3z + 2$
64. $-5u^2 - 13u + 6$

In Exercises 65–72, factor by grouping.

65. $x^3 - x^2 + 2x - 2$
66. $x^3 + 5x^2 - 5x - 25$
67. $2x^3 - x^2 - 6x + 3$
68. $5x^3 - 10x^2 + 3x - 6$
69. $6 + 2x - 3x^3 - x^4$
70. $x^5 + 2x^3 + x^2 + 2$
71. $6x^3 - 2x + 3x^2 - 1$
72. $8x^5 - 6x^2 + 12x^3 - 9$

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In Exercises 73–78, factor the trinomial by grouping.

73. $3x^2 + 10x + 8$

74. $2x^2 + 9x + 9$

75. $6x^2 + x - 2$

76. $6x^2 - x - 15$

77. $15x^2 - 11x + 2$

78. $12x^2 - 13x + 1$

In Exercises 79–112, completely factor the expression.

79. $6x^2 - 54$

80. $12x^2 - 48$

81. $x^3 - 4x^2$

82. $x^3 - 9x$

83. $x^2 - 2x + 1$

84. $16 + 6x - x^2$

85. $1 - 4x + 4x^2$

86. $-9x^2 + 6x - 1$

87. $2x^2 + 4x - 2x^3$

88. $2y^3 - 7y^2 - 15y$

89. $9x^2 + 10x + 1$

90. $13x + 6 + 5x^2$

91. $\frac{1}{81}x^2 + \frac{2}{9}x - 8$

92. $\frac{1}{8}x^2 - \frac{1}{96}x - \frac{1}{16}$

93. $3x^3 + x^2 + 15x + 5$

94. $5 - x + 5x^2 - x^3$

95. $x^4 - 4x^3 + x^2 - 4x$

96. $3u - 2u^2 + 6 - u^3$

97. $\frac{1}{4}x^3 + 3x^2 + \frac{3}{4}x + 9$

98. $\frac{1}{5}x^3 + x^2 - x - 5$

99. $(t - 1)^2 - 49$

100. $(x^2 + 1)^2 - 4x^2$

101. $(x^2 + 8)^2 - 36x^2$

102. $2t^3 - 16$

103. $5x^3 + 40$

104. $4x(2x - 1) + (2x - 1)^2$

105. $5(3 - 4x)^2 - 8(3 - 4x)(5x - 1)$

106. $2(x + 1)(x - 3)^2 - 3(x + 1)^2(x - 3)$

107. $7(3x + 2)^2(1 - x)^2 + (3x + 2)(1 - x)^3$

108. $7x(2)(x^2 + 1)(2x) - (x^2 + 1)^2(7)$

109. $3(x - 2)^2(x + 1)^4 + (x - 2)^3(4)(x + 1)^3$

110. $2x(x - 5)^4 - x^2(4)(x - 5)^3$

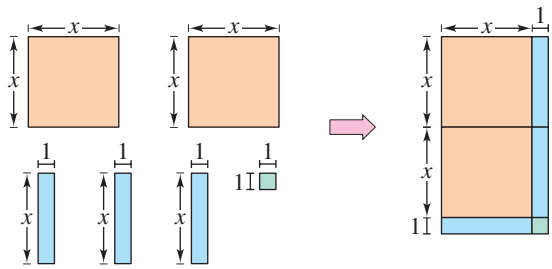
111. $5(x^6 + 1)^4(6x^5)(3x + 2)^3 + 3(3x + 2)^2(3)(x^6 + 1)^5$

112. $\frac{x^2}{2}(x^2 + 1)^4 - (x^2 + 1)^5$

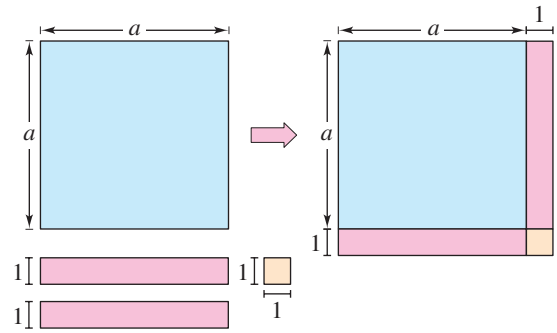
Geometric Modeling **In Exercises 113–116, match the factoring formula with the correct "geometric factoring model." [The models are labeled (a), (b), (c), and (d).] For instance, a factoring model for**

$$2x^2 + 3x + 1 = (2x + 1)(x + 1)$$

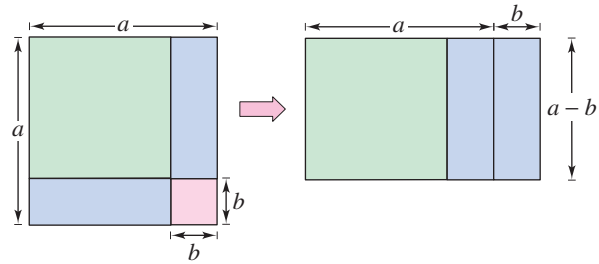
is shown in the figure.



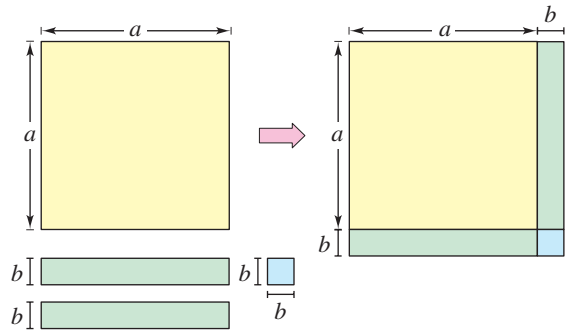
(a)



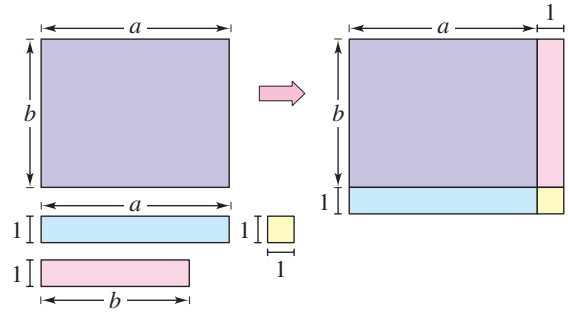
(b)



(c)



(d)



113. $a^2 - b^2 = (a + b)(a - b)$

114. $a^2 + 2ab + b^2 = (a + b)^2$

115. $a^2 + 2a + 1 = (a + 1)^2$

116. $ab + a + b + 1 = (a + 1)(b + 1)$

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Geometric Modeling **In Exercises 117–120, draw a "geometric factoring model" to represent the factorization.**

117. $3x^2 + 7x + 2 = (3x + 1)(x + 2)$

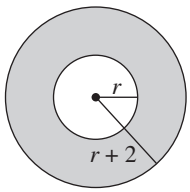
118. $x^2 + 4x + 3 = (x + 3)(x + 1)$

119. $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

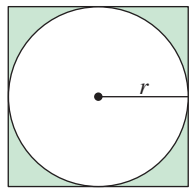
120. $x^2 + 3x + 2 = (x + 2)(x + 1)$

Geometry **In Exercises 121–124, write an expression in factored form for the area of the shaded portion of the figure.**

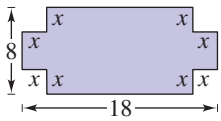
121.



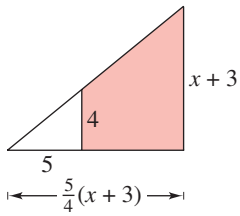
122.



123.



124.



In Exercises 125–128, find all values of *b* for which the trinomial can be factored.

125. $x^2 + bx - 15$

126. $x^2 + bx + 50$

127. $x^2 + bx - 12$

128. $x^2 + bx + 24$

In Exercises 129–132, find two integer values of *c* such that the trinomial can be factored. (There are many correct answers.)

129. $2x^2 + 5x + c$

130. $3x^2 - 10x + c$

131. $3x^2 - x + c$

132. $2x^2 + 9x + c$

133. ***Error Analysis*** Describe the error.

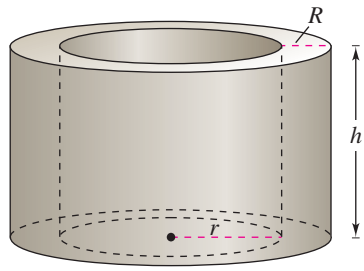
$$9x^2 - 9x - 54 = (3x + 6)(3x - 9)$$
$$= 3(x + 2)(x - 3)$$

134. ***Think About It*** Is $(3x - 6)(x + 1)$ completely factored? Explain.

135. ***Chemistry*** The rate of change of an autocatalytic chemical reaction is $kQx - kx^2$, where Q is the amount of the original substance, x is the amount of substance formed, and k is a constant of proportionality. Factor the expression.

Model It

136. ***Geometry*** The volume V of concrete used to make the cylindrical concrete storage tank shown in the figure is $V = \pi R^2 h - \pi r^2 h$ where R is the outside radius, r is the inside radius, and h is the height of the storage tank.



(a) Factor the expression for the volume.

(b) From the result of part (a), show that the volume of concrete is

2π(average radius)(thickness of the tank)h.

(c) An 80-pound bag of concrete mix yields $\frac{3}{5}$ cubic foot of concrete. Find the number of bags required to construct a concrete storage tank having the following dimensions.

Outside radius, $R = 4$ feet

Inside radius, $r = 3\frac{2}{3}$ feet

Height, h feet

(d) Use the *table* feature of a graphing utility to create a table showing the number of bags of concrete required to construct the storage tank in part (c) with heights of $h = \frac{1}{2}, h = 1, h = \frac{3}{2}, h = 2, \ldots,$ $h = 6$ feet.

Synthesis

True or False? **In Exercises 137 and 138, determine whether the statement is true or false. Justify your answer.**

137. The difference of two perfect squares can be factored as the product of conjugate pairs.

138. The sum of two perfect squares can be factored as the binomial sum squared.

139. Factor $x^{2n} - y^{2n}$ completely.

140. Factor $x^{3n} + y^{3n}$ completely.

141. Factor $x^{3n} - y^{2n}$ completely.

142. ***Writing*** Explain what is meant when it is said that a polynomial is in factored form.

143. Give an example of a polynomial that is prime with respect to the integers.

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P.5 Rational Expressions

What you should learn

- Find domains of algebraic expressions.
- Simplify rational expressions.
- Add, subtract, multiply, and divide rational expressions.
- Simplify complex fractions and rewrite difference quotients.

Why you should learn it

Rational expressions can be used to solve real-life problems. For instance, in Exercise 84 on page 50, a rational expression is used to model the projected number of households banking and paying bills online from 2002 through 2007.



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Domain of an Algebraic Expression

The set of real numbers for which an algebraic expression is defined is the **domain** of the expression. Two algebraic expressions are **equivalent** if they have the same domain and yield the same values for all numbers in their domain. For instance, $(x + 1) + (x + 2)$ and $2x + 3$ are equivalent because

$$\begin{aligned}(x + 1) + (x + 2) &= x + 1 + x + 2\\ &= x + x + 1 + 2\\ &= 2x + 3.\end{aligned}$$

Example 1 **Finding the Domain of an Algebraic Expression**

a. The domain of the polynomial

$$2x^3 + 3x + 4$$

is the set of all real numbers. In fact, the domain of any polynomial is the set of all real numbers, unless the domain is specifically restricted.

b. The domain of the radical expression

$$\sqrt{x - 2}$$

is the set of real numbers greater than or equal to 2, because the square root of a negative number is not a real number.

c. The domain of the expression

$$\frac{x + 2}{x - 3}$$

is the set of all real numbers except $x = 3$, which would result in division by zero, which is undefined.

CHECKPOINT Now try Exercise 1.

The quotient of two algebraic expressions is a *fractional expression*. Moreover, the quotient of two *polynomials* such as

$$\frac{1}{x}, \qquad \frac{2x - 1}{x + 1}, \qquad \text{or} \qquad \frac{x^2 - 1}{x^2 + 1}$$

is a **rational expression.** Recall that a fraction is in simplest form if its numerator and denominator have no factors in common aside from ± 1. To write a fraction in simplest form, divide out common factors.

$$\frac{a \cdot \cancel{c}}{b \cdot \cancel{c}} = \frac{a}{b}, \quad c \neq 0$$

The key to success in simplifying rational expressions lies in your ability to *factor* polynomials.

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Simplifying Rational Expressions

When simplifying rational expressions, be sure to factor each polynomial completely before concluding that the numerator and denominator have no factors in common.

In this text, when a rational expression is written, the domain is usually not listed with the expression. It is *implied* that the real numbers that make the denominator zero are excluded from the expression. Also, when performing operations with rational expressions, this text follows the convention of listing *by the simplified expression* all values of x that must be specifically excluded from the domain in order to make the domains of the simplified and original expressions agree.

Example 2 **Simplifying a Rational Expression**

Write $\dfrac{x^2 + 4x - 12}{3x - 6}$ in simplest form.

Solution

$$\frac{x^2 + 4x - 12}{3x - 6} = \frac{(x + 6)(x - 2)}{3(x - 2)}$$ Factor completely.

$$= \frac{x + 6}{3}, \quad x \neq 2$$ Divide out common factors.

Note that the original expression is undefined when $x = 2$ (because division by zero is undefined). To make sure that the simplified expression is *equivalent* to the original expression, you must restrict the domain of the simplified expression by excluding the value $x = 2$.

CHECKPOINT Now try Exercise 19.

STUDY TIP

In Example 2, do not make the mistake of trying to simplify further by dividing out terms.

$$\frac{x + 6}{3} = \frac{x + 6}{3} = x + 2$$

Remember that to simplify fractions, divide out common *factors*, not terms.

Sometimes it may be necessary to change the sign of a factor to simplify a rational expression, as shown in Example 3.

Example 3 **Simplifying Rational Expressions**

Write $\dfrac{12 + x - x^2}{2x^2 - 9x + 4}$ in simplest form.

Solution

$$\frac{12 + x - x^2}{2x^2 - 9x + 4} = \frac{(4 - x)(3 + x)}{(2x - 1)(x - 4)}$$ Factor completely.

$$= \frac{-(x - 4)(3 + x)}{(2x - 1)(x - 4)}$$ $(4 - x) = -(x - 4)$

$$= -\frac{3 + x}{2x - 1}, \quad x \neq 4$$ Divide out common factors.

CHECKPOINT Now try Exercise 25.

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Operations with Rational Expressions

To multiply or divide rational expressions, use the properties of fractions discussed in Section P.1. Recall that to divide fractions, you invert the divisor and multiply.

Example 4 Multiplying Rational Expressions

$$\frac{2x^2 + x - 6}{x^2 + 4x - 5} \cdot \frac{x^3 - 3x^2 + 2x}{4x^2 - 6x} = \frac{(2x - 3)(x + 2)}{(x + 5)(x - 1)} \cdot \frac{x(x - 2)(x - 1)}{2x(2x - 3)}$$

$$= \frac{(x + 2)(x - 2)}{2(x + 5)}, \qquad x \neq 0, x \neq 1, x \neq \tfrac{3}{2}$$

CHECKPOINT Now try Exercise 39.

In Example 4 the restrictions $x \neq 0$, $x \neq 1$, and $x \neq \frac{3}{2}$ are listed with the simplified expression in order to make the two domains agree. Note that the value $x = -5$ is excluded from both domains, so it is not necessary to list this value.

Example 5 Dividing Rational Expressions

$$\frac{x^3 - 8}{x^2 - 4} \div \frac{x^2 + 2x + 4}{x^3 + 8} = \frac{x^3 - 8}{x^2 - 4} \cdot \frac{x^3 + 8}{x^2 + 2x + 4}$$ Invert and multiply.

$$= \frac{(x - 2)(x^2 + 2x + 4)}{(x + 2)(x - 2)} \cdot \frac{(x + 2)(x^2 - 2x + 4)}{(x^2 + 2x + 4)}$$

$$= x^2 - 2x + 4, \qquad x \neq \pm 2$$ Divide out common factors.

CHECKPOINT Now try Exercise 41.

To add or subtract rational expressions, you can use the LCD (least common denominator) method or the *basic definition*

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}, \qquad b \neq 0, d \neq 0.$$ Basic definition

This definition provides an efficient way of adding or subtracting *two* fractions that have no common factors in their denominators.

Example 6 Subtracting Rational Expressions

$$\frac{x}{x - 3} - \frac{2}{3x + 4} = \frac{x(3x + 4) - 2(x - 3)}{(x - 3)(3x + 4)}$$ Basic definition

$$= \frac{3x^2 + 4x - 2x + 6}{(x - 3)(3x + 4)}$$ Distributive Property

$$= \frac{3x^2 + 2x + 6}{(x - 3)(3x + 4)}$$ Combine like terms.

CHECKPOINT Now try Exercise 49.

STUDY TIP

When subtracting rational expressions, remember to distribute the negative sign to all the terms in the quantity that is being subtracted.

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For three or more fractions, or for fractions with a repeated factor in the denominators, the LCD method works well. Recall that the least common denominator of several fractions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator. Here is a numerical example.

$$\frac{1}{6}+\frac{3}{4}-\frac{2}{3}=\frac{1\cdot 2}{6\cdot 2}+\frac{3\cdot 3}{4\cdot 3}-\frac{2\cdot 4}{3\cdot 4}$$ The LCD is 12.

$$=\frac{2}{12}+\frac{9}{12}-\frac{8}{12}$$

$$=\frac{3}{12}$$

$$=\frac{1}{4}$$

Sometimes the numerator of the answer has a factor in common with the denominator. In such cases the answer should be simplified. For instance, in the example above, $\frac{3}{12}$ was simplified to $\frac{1}{4}$.

Example 7 **Combining Rational Expressions: The LCD Method**

Perform the operations and simplify.

$$\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{x^2-1}$$

Solution

Using the factored denominators $(x-1)$, x, and $(x+1)(x-1)$, you can see that the LCD is $x(x+1)(x-1)$.

$$\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{(x+1)(x-1)}$$

$$=\frac{3(x)(x+1)}{x(x+1)(x-1)}-\frac{2(x+1)(x-1)}{x(x+1)(x-1)}+\frac{(x+3)(x)}{x(x+1)(x-1)}$$

$$=\frac{3(x)(x+1)-2(x+1)(x-1)+(x+3)(x)}{x(x+1)(x-1)}$$

$$=\frac{3x^2+3x-2x^2+2+x^2+3x}{x(x+1)(x-1)}$$ Distributive Property

$$=\frac{3x^2-2x^2+x^2+3x+3x+2}{x(x+1)(x-1)}$$ Group like terms.

$$=\frac{2x^2+6x+2}{x(x+1)(x-1)}$$ Combine like terms.

$$=\frac{2(x^2+3x+1)}{x(x+1)(x-1)}$$ Factor.

CHECKPOINT Now try Exercise 51.

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Complex Fractions and the Difference Quotient

Fractional expressions with separate fractions in the numerator, denominator, or both are called **complex fractions.** Here are two examples.

$$\frac{\left(\frac{1}{x}\right)}{x^2+1} \quad \text{and} \quad \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x^2+1}\right)}$$

A complex fraction can be simplified by combining the fractions in its numerator into a single fraction and then combining the fractions in its denominator into a single fraction. Then invert the denominator and multiply.

Example 8 **Simplifying a Complex Fraction**

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left[\frac{2-3(x)}{x}\right]}{\left[\frac{1(x-1)-1}{x-1}\right]}$$ Combine fractions.

$$= \frac{\left(\frac{2-3x}{x}\right)}{\left(\frac{x-2}{x-1}\right)}$$ Simplify.

$$= \frac{2-3x}{x} \cdot \frac{x-1}{x-2}$$ Invert and multiply.

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x \neq 1$$

CHECKPOINT Now try Exercise 57.

Another way to simplify a complex fraction is to multiply its numerator and denominator by the LCD of all fractions in its numerator and denominator. This method is applied to the fraction in Example 8 as follows.

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} \cdot \frac{x(x-1)}{x(x-1)}$$ LCD is $x(x-1)$.

$$= \frac{\left(\frac{2-3x}{\cancel{x}}\right) \cdot \cancel{x}(x-1)}{\left(\frac{x-2}{\cancel{x-1}}\right) \cdot x(\cancel{x-1})}$$

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x \neq 1$$

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The next three examples illustrate some methods for simplifying rational expressions involving negative exponents and radicals. These types of expressions occur frequently in calculus.

To simplify an expression with negative exponents, one method is to begin by factoring out the common factor with the *smaller* exponent. Remember that when factoring, you *subtract* exponents. For instance, in $3x^{-5/2} + 2x^{-3/2}$ the smaller exponent is $-\frac{5}{2}$ and the common factor is $x^{-5/2}$.

$$\begin{aligned} 3x^{-5/2} + 2x^{-3/2} &= x^{-5/2}[3(1) + 2x^{-3/2-(-5/2)}] \\ &= x^{-5/2}(3 + 2x^1) \\ &= \frac{3 + 2x}{x^{5/2}} \end{aligned}$$

Example 9 **Simplifying an Expression**

Simplify the following expression containing negative exponents.

$$x(1 - 2x)^{-3/2} + (1 - 2x)^{-1/2}$$

Solution

Begin by factoring out the common factor with the *smaller exponent*.

$$\begin{aligned} x(1 - 2x)^{-3/2} + (1 - 2x)^{-1/2} &= (1 - 2x)^{-3/2}[x + (1 - 2x)^{(-1/2)-(-3/2)}] \\ &= (1 - 2x)^{-3/2}[x + (1 - 2x)^1] \\ &= \frac{1 - x}{(1 - 2x)^{3/2}} \end{aligned}$$

CHECKPOINT Now try Exercise 65.

A second method for simplifying an expression with negative exponents is shown in the next example.

Example 10 **Simplifying an Expression with Negative Exponents**

$$\begin{aligned} &\frac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2} \\ &\quad = \frac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2} \cdot \frac{(4 - x^2)^{1/2}}{(4 - x^2)^{1/2}} \\ &\quad = \frac{(4 - x^2)^1 + x^2(4 - x^2)^0}{(4 - x^2)^{3/2}} \\ &\quad = \frac{4 - x^2 + x^2}{(4 - x^2)^{3/2}} \\ &\quad = \frac{4}{(4 - x^2)^{3/2}} \end{aligned}$$

CHECKPOINT Now try Exercise 67.

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Example 11 **Rewriting a Difference Quotient**

The following expression from calculus is an example of a *difference quotient*.

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Rewrite this expression by rationalizing its numerator.

Solution

$$\begin{aligned}\frac{\sqrt{x+h}-\sqrt{x}}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\ &= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}, \qquad h\neq 0\end{aligned}$$

Notice that the original expression is undefined when $h = 0$. So, you must exclude $h = 0$ from the domain of the simplified expression so that the expressions are equivalent.

CHECKPOINT Now try Exercise 73.

Difference quotients, such as that in Example 11, occur frequently in calculus. Often, they need to be rewritten in an equivalent form that can be evaluated when $h = 0$. Note that the equivalent form is not simpler than the original form, but it has the advantage that it is defined when $h = 0$.

P.5 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. The set of real numbers for which an algebraic expression is defined is the _____ of the expression.
2. The quotient of two algebraic expressions is a fractional expression and the quotient of two polynomials is a _____ _____.
3. Fractional expressions with separate fractions in the numerator, denominator, or both are called _____ fractions.
4. To simplify an expression with negative exponents, it is possible to begin by factoring out the common factor with the _____ exponent.
5. Two algebraic expressions that have the same domain and yield the same values for all numbers in their domains are called _____.
6. An important rational expression, such as $\frac{(x+h)^2-x^2}{h}$, that occurs in calculus is called a _____ _____.

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In Exercises 1–8, find the domain of the expression.

1. $3x^2 - 4x + 7$

2. $2x^2 + 5x - 2$

3. $4x^3 + 3, \quad x \geq 0$

4. $6x^2 - 9, \quad x > 0$

5. $\dfrac{1}{x - 2}$

6. $\dfrac{x + 1}{2x + 1}$

7. $\sqrt{x + 1}$

8. $\sqrt{6 - x}$

In Exercises 9 and 10, find the missing factor in the numerator such that the two fractions are equivalent.

9. $\dfrac{5}{2x} = \dfrac{5(\quad)}{6x^2}$

10. $\dfrac{3}{4} = \dfrac{3(\quad)}{4(x + 1)}$

In Exercises 11–28, write the rational expression in simplest form.

11. $\dfrac{15x^2}{10x}$

12. $\dfrac{18y^2}{60y^5}$

13. $\dfrac{3xy}{xy + x}$

14. $\dfrac{2x^2y}{xy - y}$

15. $\dfrac{4y - 8y^2}{10y - 5}$

16. $\dfrac{9x^2 + 9x}{2x + 2}$

17. $\dfrac{x - 5}{10 - 2x}$

18. $\dfrac{12 - 4x}{x - 3}$

19. $\dfrac{y^2 - 16}{y + 4}$

20. $\dfrac{x^2 - 25}{5 - x}$

21. $\dfrac{x^3 + 5x^2 + 6x}{x^2 - 4}$

22. $\dfrac{x^2 + 8x - 20}{x^2 + 11x + 10}$

23. $\dfrac{y^2 - 7y + 12}{y^2 + 3y - 18}$

24. $\dfrac{x^2 - 7x + 6}{x^2 + 11x + 10}$

25. $\dfrac{2 - x + 2x^2 - x^3}{x^2 - 4}$

26. $\dfrac{x^2 - 9}{x^3 + x^2 - 9x - 9}$

27. $\dfrac{z^3 - 8}{z^2 + 2z + 4}$

28. $\dfrac{y^3 - 2y^2 - 3y}{y^3 + 1}$

In Exercises 29 and 30, complete the table. What can you conclude?

29.

x	0	1	2	3	4	5	6
$\dfrac{x^2 - 2x - 3}{x - 3}$							
$x + 1$							

30.

x	0	1	2	3	4	5	6
$\dfrac{x - 3}{x^2 - x - 6}$							
$\dfrac{1}{x + 2}$							

31. ***Error Analysis*** Describe the error.

$$\frac{5x^3}{2x^3 + 4} = \frac{5x^3}{2x^3 + 4} = \frac{5}{2 + 4} = \frac{5}{6}$$

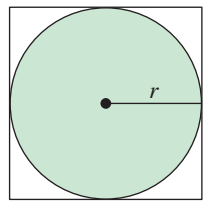
32. ***Error Analysis*** Describe the error.

$$\frac{x^3 + 25x}{x^2 - 2x - 15} = \frac{x(x^2 + 25)}{(x - 5)(x + 3)}$$

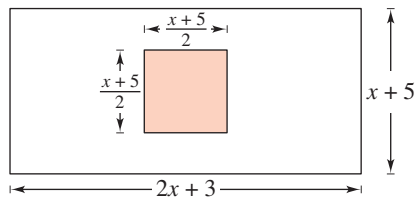
$$= \frac{x(x + 5)(x - 5)}{(x - 5)(x + 3)} = \frac{x(x + 5)}{x + 3}$$

Geometry **In Exercises 33 and 34, find the ratio of the area of the shaded portion of the figure to the total area of the figure.**

33.



34.



In Exercises 35–42, perform the multiplication or division and simplify.

35. $\dfrac{5}{x - 1} \cdot \dfrac{x - 1}{25(x - 2)}$

36. $\dfrac{x + 13}{x^3(3 - x)} \cdot \dfrac{x(x - 3)}{5}$

37. $\dfrac{r}{r - 1} \cdot \dfrac{r^2 - 1}{r^2}$

38. $\dfrac{4y - 16}{5y + 15} \cdot \dfrac{2y + 6}{4 - y}$

39. $\dfrac{t^2 - t - 6}{t^2 + 6t + 9} \cdot \dfrac{t + 3}{t^2 - 4}$

40. $\dfrac{x^2 + xy - 2y^2}{x^3 + x^2y} \cdot \dfrac{x}{x^2 + 3xy + 2y^2}$

41. $\dfrac{x^2 - 36}{x} \div \dfrac{x^3 - 6x^2}{x^2 + x}$

42. $\dfrac{x^2 - 14x + 49}{x^2 - 49} \div \dfrac{3x - 21}{x + 7}$

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In Exercises 43–52, perform the addition or subtraction and simplify.

43. $\dfrac{5}{x-1}+\dfrac{x}{x-1}$

44. $\dfrac{2x-1}{x+3}+\dfrac{1-x}{x+3}$

45. $6-\dfrac{5}{x+3}$

46. $\dfrac{3}{x-1}-5$

47. $\dfrac{3}{x-2}+\dfrac{5}{2-x}$

48. $\dfrac{2x}{x-5}-\dfrac{5}{5-x}$

49. $\dfrac{1}{x^2-x-2}-\dfrac{x}{x^2-5x+6}$

50. $\dfrac{2}{x^2-x-2}+\dfrac{10}{x^2+2x-8}$

51. $-\dfrac{1}{x}+\dfrac{2}{x^2+1}+\dfrac{1}{x^3+x}$

52. $\dfrac{2}{x+1}+\dfrac{2}{x-1}+\dfrac{1}{x^2-1}$

Error Analysis **In Exercises 53 and 54, describe the error.**

53.
$$\frac{x+4}{x+2}-\frac{3x-8}{x+2}=\frac{x+4-3x-8}{x+2}$$
$$=\frac{-2x-4}{x+2}=\frac{-2(x+2)}{x+2}=-2$$

54.
$$\frac{6-x}{x(x+2)}+\frac{x+2}{x^2}+\frac{8}{x^2(x+2)}$$
$$=\frac{x(6-x)+(x+2)^2+8}{x^2(x+2)}$$
$$=\frac{6x-x^2+x^2+4+8}{x^2(x+2)}$$
$$=\frac{6(x+2)}{x^2(x+2)}=\frac{6}{x^2}$$

In Exercises 55–60, simplify the complex fraction.

55. $\dfrac{\left(\dfrac{x}{2}-1\right)}{(x-2)}$

56. $\dfrac{(x-4)}{\left(\dfrac{x}{4}-\dfrac{4}{x}\right)}$

57. $\dfrac{\left[\dfrac{x^2}{(x+1)^2}\right]}{\left[\dfrac{x}{(x+1)^3}\right]}$

58. $\dfrac{\left(\dfrac{x^2-1}{x}\right)}{\left[\dfrac{(x-1)^2}{x}\right]}$

59. $\dfrac{\left(\sqrt{x}-\dfrac{1}{2\sqrt{x}}\right)}{\sqrt{x}}$

60. $\dfrac{\left(\dfrac{t^2}{\sqrt{t^2+1}}-\sqrt{t^2+1}\right)}{t^2}$

In Exercises 61–66, factor the expression by removing the common factor with the smaller exponent.

61. x^5-2x^{-2}

62. x^5-5x^{-3}

63. $x^2(x^2+1)^{-5}-(x^2+1)^{-4}$

64. $2x(x-5)^{-3}-4x^2(x-5)^{-4}$

65. $2x^2(x-1)^{1/2}-5(x-1)^{-1/2}$

66. $4x^3(2x-1)^{3/2}-2x(2x-1)^{-1/2}$

In Exercises 67 and 68, simplify the expression.

67. $\dfrac{3x^{1/3}-x^{-2/3}}{3x^{-2/3}}$

68. $\dfrac{-x^3(1-x^2)^{-1/2}-2x(1-x^2)^{1/2}}{x^4}$

In Exercises 69–72, simplify the difference quotient.

69. $\dfrac{\left(\dfrac{1}{x+h}-\dfrac{1}{x}\right)}{h}$

70. $\dfrac{\left[\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}\right]}{h}$

71. $\dfrac{\left(\dfrac{1}{x+h-4}-\dfrac{1}{x-4}\right)}{h}$

72. $\dfrac{\left(\dfrac{x+h}{x+h+1}-\dfrac{x}{x+1}\right)}{h}$

In Exercises 73–76, simplify the difference quotient by rationalizing the numerator.

73. $\dfrac{\sqrt{x+2}-\sqrt{x}}{2}$

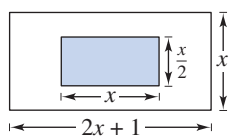
74. $\dfrac{\sqrt{z-3}-\sqrt{z}}{3}$

75. $\dfrac{\sqrt{x+h+1}-\sqrt{x+1}}{h}$

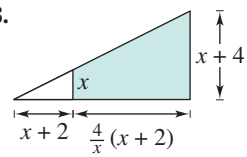
76. $\dfrac{\sqrt{x+h-2}-\sqrt{x-2}}{h}$

Probability **In Exercises 77 and 78, consider an experiment in which a marble is tossed into a box whose base is shown in the figure. The probability that the marble will come to rest in the shaded portion of the box is equal to the ratio of the shaded area to the total area of the figure. Find the probability.**

77.



78.



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79. ***Rate*** A photocopier copies at a rate of 16 pages per minute.

(a) Find the time required to copy one page.

(b) Find the time required to copy x pages.

(c) Find the time required to copy 60 pages.

80. ***Rate*** After working together for t hours on a common task, two workers have done fractional parts of the job equal to $t/3$ and $t/5$, respectively. What fractional part of the task has been completed?

Finance **In Exercises 81 and 82, the formula that approximates the annual interest rate r of a monthly installment loan is given by**

$$r = \frac{\left[\frac{24(NM - P)}{N}\right]}{\left(P + \frac{NM}{12}\right)}$$

where N is the total number of payments, M is the monthly payment, and P is the amount financed.

81. (a) Approximate the annual interest rate for a four-year car loan of \$16,000 that has monthly payments of \$400.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

82. (a) Approximate the annual interest rate for a five-year car loan of \$28,000 that has monthly payments of \$525.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

83. ***Refrigeration*** When food (at room temperature) is placed in a refrigerator, the time required for the food to cool depends on the amount of food, the air circulation in the refrigerator, the original temperature of the food, and the temperature of the refrigerator. The model that gives the temperature of food that has an original temperature of 75°F and is placed in a 40°F refrigerator is

$$T = 10\left(\frac{4t^2 + 16t + 75}{t^2 + 4t + 10}\right)$$

where T is the temperature (in degrees Fahrenheit) and t is the time (in hours).

(a) Complete the table.

t	0	2	4	6	8	10
T						

t	12	14	16	18	20	22
T						

(b) What value of T does the mathematical model appear to be approaching?

Model It

84. ***Interactive Money Management*** The table shows the projected numbers of U.S. households (in millions) banking online and paying bills online for the years 2002 through 2007. (Source: eMarketer; Forrester Research)

Year	Banking	Paying Bills
2002	21.9	13.7
2003	26.8	17.4
2004	31.5	20.9
2005	35.0	23.9
2006	40.0	26.7
2007	45.0	29.1

Mathematical models for these data are

$$\text{Number banking online} = \frac{-0.728t^2 + 23.81t - 0.3}{-0.049t^2 + 0.61t + 1.0}$$

and

$$\text{Number paying bills online} = \frac{4.39t + 5.5}{0.002t^2 + 0.01t + 1.0}$$

where t represents the year, with $t = 2$ corresponding to 2002.

(a) Using the models, create a table to estimate the projected number of households banking online and the projected number of households paying bills online for the given years.

(b) Compare the values given by the models with the actual data.

(c) Determine a model for the ratio of the projected number of households paying bills online to the projected number of households banking online.

(d) Use the model from part (c) to find the ratio over the given years. Interpret your results.

Synthesis

True or False? **In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.**

85. $\dfrac{x^{2n} - 1^{2n}}{x^n - 1^n} = x^n + 1^n$

86. $\dfrac{x^2 - 3x + 2}{x - 1} = x - 2$, for all values of x.

87. ***Think About It*** How do you determine whether a rational expression is in simplest form?

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P.6 Errors and the Algebra of Calculus

What you should learn

- Avoid common algebraic errors.
- Recognize and use algebraic techniques that are common in calculus.

Why you should learn it

An efficient command of algebra is critical in mastering this course and in the study of calculus.

Algebraic Errors to Avoid

This section contains five lists of common algebraic errors: errors involving parentheses, errors involving fractions, errors involving exponents, errors involving radicals, and errors involving dividing out. Many of these errors are made because they seem to be the *easiest* things to do. For instance, the operations of subtraction and division are often believed to be commutative and associative. The following examples illustrate the fact that subtraction and division are neither commutative nor associative.

Not commutative

$4 - 3 \neq 3 - 4$

$15 \div 5 \neq 5 \div 15$

Not associative

$8 - (6 - 2) \neq (8 - 6) - 2$

$20 \div (4 \div 2) \neq (20 \div 4) \div 2$

Errors Involving Parentheses

Potential Error	*Correct Form*	*Comment*
$\cancel{a - (x - b) = a - x - b}$	$a - (x - b) = a - x + b$	Change all signs when distributing minus sign.
$\cancel{(a + b)^2 = a^2 + b^2}$	$(a + b)^2 = a^2 + 2ab + b^2$	Remember the middle term when squaring binomials.
$\cancel{\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right) = \frac{1}{2}(ab)}$	$\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right) = \frac{1}{4}(ab) = \frac{ab}{4}$	$\frac{1}{2}$ occurs twice as a factor.
$\cancel{(3x + 6)^2 = 3(x + 2)^2}$	$(3x + 6)^2 = [3(x + 2)]^2$ $= 3^2(x + 2)^2$	When factoring, apply exponents to all factors.

Errors Involving Fractions

Potential Error	*Correct Form*	*Comment*
$\cancel{\frac{a}{x + b} = \frac{a}{x} + \frac{a}{b}}$	Leave as $\frac{a}{x + b}$.	Do not add denominators when adding fractions.
$\cancel{\frac{\left(\frac{x}{a}\right)}{b} = \frac{bx}{a}}$	$\frac{\left(\frac{x}{a}\right)}{b} = \left(\frac{x}{a}\right)\left(\frac{1}{b}\right) = \frac{x}{ab}$	Multiply by the reciprocal when dividing fractions.
$\cancel{\frac{1}{a} + \frac{1}{b} = \frac{1}{a + b}}$	$\frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab}$	Use the property for adding fractions.
$\cancel{\frac{1}{3x} = \frac{1}{3}x}$	$\frac{1}{3x} = \frac{1}{3} \cdot \frac{1}{x}$	Use the property for multiplying fractions.
$\cancel{(1/3)x = \frac{1}{3x}}$	$(1/3)x = \frac{1}{3} \cdot x = \frac{x}{3}$	Be careful when using a slash to denote division.
$\cancel{(1/x) + 2 = \frac{1}{x + 2}}$	$(1/x) + 2 = \frac{1}{x} + 2 = \frac{1 + 2x}{x}$	Be careful when using a slash to denote division and be sure to find a common denominator before you add fractions.

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Errors Involving Exponents

Potential Error	*Correct Form*	*Comment*
$\cancel{(x^2)^3 = x^5}$	$(x^2)^3 = x^{2\cdot 3} = x^6$	Multiply exponents when raising a power to a power.
$\cancel{x^2 \cdot x^3 = x^6}$	$x^2 \cdot x^3 = x^{2+3} = x^5$	Add exponents when multiplying powers with like bases.
$\cancel{2x^3 = (2x)^3}$	$2x^3 = 2(x^3)$	Exponents have priority over coefficients.
$\cancel{\frac{1}{x^2 - x^3} = x^{-2} - x^{-3}}$	Leave as $\frac{1}{x^2 - x^3}$.	Do not move term-by-term from denominator to numerator.

Errors Involving Radicals

Potential Error	*Correct Form*	*Comment*
$\cancel{\sqrt{5x} = 5\sqrt{x}}$	$\sqrt{5x} = \sqrt{5}\sqrt{x}$	Radicals apply to every factor inside the radical.
$\cancel{\sqrt{x^2 + a^2} = x + a}$	Leave as $\sqrt{x^2 + a^2}$.	Do not apply radicals term-by-term.
$\cancel{\sqrt{-x + a} = -\sqrt{x - a}}$	Leave as $\sqrt{-x + a}$.	Do not factor minus signs out of square roots.

Errors Involving Dividing Out

Potential Error	*Correct Form*	*Comment*
$\cancel{\frac{a + bx}{a} = 1 + bx}$	$\frac{a + bx}{a} = \frac{a}{a} + \frac{bx}{a} = 1 + \frac{b}{a}x$	Divide out common factors, not common terms.
$\cancel{\frac{a + ax}{a} = a + x}$	$\frac{a + ax}{a} = \frac{a(1 + x)}{a} = 1 + x$	Factor before dividing out.
$\cancel{1 + \frac{x}{2x} = 1 + \frac{1}{x}}$	$1 + \frac{x}{2x} = 1 + \frac{1}{2} = \frac{3}{2}$	Divide out common factors.

A good way to avoid errors is to *work slowly*, *write neatly*, and *talk to yourself*. Each time you write a step, ask yourself why the step is algebraically legitimate. You can justify the step below because *dividing the numerator and denominator by the same nonzero number produces an equivalent fraction*.

$$\frac{2x}{6} = \frac{\cancel{2} \cdot x}{\cancel{2} \cdot 3} = \frac{x}{3}$$

Example 1 Using the Property for Adding Fractions

Describe and correct the error. $\cancel{\frac{1}{2x} + \frac{1}{3x} = \frac{1}{5x}}$

Solution

When adding fractions, use the property for adding fractions: $\frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab}$.

$$\frac{1}{2x} + \frac{1}{3x} = \frac{3x + 2x}{6x^2} = \frac{5x}{6x^2} = \frac{5}{6x}$$

CHECKPOINT Now try Exercise 17.

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Some Algebra of Calculus

In calculus it is often necessary to take a simplified algebraic expression and "unsimplify" it. See the following lists, taken from a standard calculus text.

Unusual Factoring

Expression	*Useful Calculus Form*	*Comment*
$\frac{5x^4}{8}$	$\frac{5}{8}x^4$	Write with fractional coefficient.
$\frac{x^2 + 3x}{-6}$	$-\frac{1}{6}(x^2 + 3x)$	Write with fractional coefficient.
$2x^2 - x - 3$	$2\left(x^2 - \frac{x}{2} - \frac{3}{2}\right)$	Factor out the leading coefficient.
$\frac{x}{2}(x + 1)^{-1/2} + (x + 1)^{1/2}$	$\frac{(x + 1)^{-1/2}}{2}[x + 2(x + 1)]$	Factor out factor with lowest power.

Writing with Negative Exponents

Expression	*Useful Calculus Form*	*Comment*
$\frac{9}{5x^3}$	$\frac{9}{5}x^{-3}$	Move the factor to the numerator and change the sign of the exponent.
$\frac{7}{\sqrt{2x - 3}}$	$7(2x - 3)^{-1/2}$	Move the factor to the numerator and change the sign of the exponent.

Writing a Fraction as a Sum

Expression	*Useful Calculus Form*	*Comment*
$\frac{x + 2x^2 + 1}{\sqrt{x}}$	$x^{1/2} + 2x^{3/2} + x^{-1/2}$	Divide each term by $x^{1/2}$.
$\frac{1 + x}{x^2 + 1}$	$\frac{1}{x^2 + 1} + \frac{x}{x^2 + 1}$	Rewrite the fraction as the sum of fractions.
$\frac{2x}{x^2 + 2x + 1}$	$\frac{2x + 2 - 2}{x^2 + 2x + 1}$	Add and subtract the same term.
	$= \frac{2x + 2}{x^2 + 2x + 1} - \frac{2}{(x + 1)^2}$	Rewrite the fraction as the difference of fractions.
$\frac{x^2 - 2}{x + 1}$	$x - 1 - \frac{1}{x + 1}$	Use long division. (See Section 3.3.)
$\frac{x + 7}{x^2 - x - 6}$	$\frac{2}{x - 3} - \frac{1}{x + 2}$	Use the method of partial fractions. (See Section 6.4.)

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Inserting Factors and Terms

Expression	*Useful Calculus Form*	*Comment*
$(2x - 1)^3$	$\frac{1}{2}(2x - 1)^3(2)$	Multiply and divide by 2.
$7x^2(4x^3 - 5)^{1/2}$	$\frac{7}{12}(4x^3 - 5)^{1/2}(12x^2)$	Multiply and divide by 12.
$\frac{4x^2}{9} - 4y^2 = 1$	$\frac{x^2}{9/4} - \frac{y^2}{1/4} = 1$	Write with fractional denominators.
$\frac{x}{x + 1}$	$\frac{x + 1 - 1}{x + 1} = 1 - \frac{1}{x + 1}$	Add and subtract the same term.

The next five examples demonstrate many of the steps in the preceding lists.

Example 2 Factors Involving Negative Exponents

Factor $x(x + 1)^{-1/2} + (x + 1)^{1/2}$.

Solution

When multiplying factors with like bases, you add exponents. When factoring, you are undoing multiplication, and so you *subtract* exponents.

$$\begin{aligned} x(x + 1)^{-1/2} + (x + 1)^{1/2} &= (x + 1)^{-1/2}[x(x + 1)^0 + (x + 1)^1] \\ &= (x + 1)^{-1/2}[x + (x + 1)] \\ &= (x + 1)^{-1/2}(2x + 1) \end{aligned}$$

CHECKPOINT Now try Exercise 23.

Another way to simplify the expression in Example 2 is to multiply the expression by a fractional form of 1 and then use the Distributive Property.

$$\begin{aligned} x(x + 1)^{-1/2} + (x + 1)^{1/2} &= [x(x + 1)^{-1/2} + (x + 1)^{1/2}] \cdot \frac{(x + 1)^{1/2}}{(x + 1)^{1/2}} \\ &= \frac{x(x + 1)^0 + (x + 1)^1}{(x + 1)^{1/2}} = \frac{2x + 1}{\sqrt{x + 1}} \end{aligned}$$

Example 3 Inserting Factors in an Expression

Insert the required factor: $\frac{x + 2}{(x^2 + 4x - 3)^2} = (\quad)\frac{1}{(x^2 + 4x - 3)^2}(2x + 4)$.

Solution

The expression on the right side of the equation is twice the expression on the left side. To make both sides equal, insert a factor of $\frac{1}{2}$.

$$\frac{x + 2}{(x^2 + 4x - 3)^2} = \left(\frac{1}{2}\right)\frac{1}{(x^2 + 4x - 3)^2}(2x + 4)$$

Right side is multiplied and divided by 2.

CHECKPOINT Now try Exercise 25.

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Example 4 **Rewriting Fractions**

Explain why the two expressions are equivalent.

$$\frac{4x^2}{9} - 4y^2 = \frac{x^2}{\frac{9}{4}} - \frac{y^2}{\frac{1}{4}}$$

Solution

To write the expression on the left side of the equation in the form given on the right side, multiply the numerators and denominators of both terms by $\frac{1}{4}$.

$$\frac{4x^2}{9} - 4y^2 = \frac{4x^2}{9}\left(\frac{\frac{1}{4}}{\frac{1}{4}}\right) - 4y^2\left(\frac{\frac{1}{4}}{\frac{1}{4}}\right) = \frac{x^2}{\frac{9}{4}} - \frac{y^2}{\frac{1}{4}}$$

CHECKPOINT Now try Exercise 29.

Example 5 **Rewriting with Negative Exponents**

Rewrite each expression using negative exponents.

a. $\dfrac{-4x}{(1 - 2x^2)^2}$ **b.** $\dfrac{2}{5x^3} - \dfrac{1}{\sqrt{x}} + \dfrac{3}{5(4x)^2}$

Solution

a. $\dfrac{-4x}{(1 - 2x^2)^2} = -4x(1 - 2x^2)^{-2}$

b. Begin by writing the second term in exponential form.

$$\frac{2}{5x^3} - \frac{1}{\sqrt{x}} + \frac{3}{5(4x)^2} = \frac{2}{5x^3} - \frac{1}{x^{1/2}} + \frac{3}{5(4x)^2}$$

$$= \frac{2}{5}x^{-3} - x^{-1/2} + \frac{3}{5}(4x)^{-2}$$

CHECKPOINT Now try Exercise 39.

Example 6 **Writing a Fraction as a Sum of Terms**

Rewrite each fraction as the sum of three terms.

a. $\dfrac{x^2 - 4x + 8}{2x}$ **b.** $\dfrac{x + 2x^2 + 1}{\sqrt{x}}$

Solution

a. $\dfrac{x^2 - 4x + 8}{2x} = \dfrac{x^2}{2x} - \dfrac{4x}{2x} + \dfrac{8}{2x}$

$$= \frac{x}{2} - 2 + \frac{4}{x}$$

b. $\dfrac{x + 2x^2 + 1}{\sqrt{x}} = \dfrac{x}{x^{1/2}} + \dfrac{2x^2}{x^{1/2}} + \dfrac{1}{x^{1/2}}$

$$= x^{1/2} + 2x^{3/2} + x^{-1/2}$$

CHECKPOINT Now try Exercise 43.

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P.6 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. To write the expression $\dfrac{2}{\sqrt{x}}$ with negative exponents, move $\sqrt{x}$ to the _____ and change the sign of the exponent.

2. When dividing fractions, multiply by the _____.

In Exercises 1–18, describe and correct the error.

1. $2x - (3y + 4) = 2x - 3y + 4$

2. $5z + 3(x - 2) = 5z + 3x - 2$

3. $\dfrac{4}{16x - (2x + 1)} = \dfrac{4}{14x + 1}$

4. $\dfrac{1 - x}{(5 - x)(-x)} = \dfrac{x - 1}{x(x - 5)}$

5. $(5z)(6z) = 30z$

6. $x(yz) = (xy)(xz)$

7. $a\left(\dfrac{x}{y}\right) = \dfrac{ax}{ay}$

8. $(4x)^2 = 4x^2$

9. $\sqrt{x + 9} = \sqrt{x} + 3$

10. $\sqrt{25 - x^2} = 5 - x$

11. $\dfrac{2x^2 + 1}{5x} = \dfrac{2x + 1}{5}$

12. $\dfrac{6x + y}{6x - y} = \dfrac{x + y}{x - y}$

13. $\dfrac{1}{a^{-1} + b^{-1}} = \left(\dfrac{1}{a + b}\right)^{-1}$

14. $\dfrac{1}{x + y^{-1}} = \dfrac{y}{x + 1}$

15. $(x^2 + 5x)^{1/2} = x(x + 5)^{1/2}$

16. $x(2x - 1)^2 = (2x^2 - x)^2$

17. $\dfrac{3}{x} + \dfrac{4}{y} = \dfrac{7}{x + y}$

18. $\dfrac{1}{2y} = (1/2)y$

In Exercises 19–38, insert the required factor in the parentheses.

19. $\dfrac{3x + 2}{5} = \dfrac{1}{5}(\quad)$

20. $\dfrac{7x^2}{10} = \dfrac{7}{10}(\quad)$

21. $\frac{2}{3}x^2 + \frac{1}{3}x + 5 = \frac{1}{3}(\quad)$

22. $\frac{3}{4}x + \frac{1}{2} = \frac{1}{4}(\quad)$

23. $x^2(x^3 - 1)^4 = (\quad)(x^3 - 1)^4(3x^2)$

24. $x(1 - 2x^2)^3 = (\quad)(1 - 2x^2)^3(-4x)$

25. $\dfrac{4x + 6}{(x^2 + 3x + 7)^3} = (\quad)\dfrac{1}{(x^2 + 3x + 7)^3}(2x + 3)$

26. $\dfrac{x + 1}{(x^2 + 2x - 3)^2} = (\quad)\dfrac{1}{(x^2 + 2x - 3)^2}(2x + 2)$

27. $\dfrac{3}{x} + \dfrac{5}{2x^2} - \dfrac{3}{2}x = (\quad)(6x + 5 - 3x^3)$

28. $\dfrac{(x - 1)^2}{169} + (y + 5)^2 = \dfrac{(x - 1)^3}{169(\quad)} + (y + 5)^2$

29. $\dfrac{9x^2}{25} + \dfrac{16y^2}{49} = \dfrac{x^2}{(\quad)} + \dfrac{y^2}{(\quad)}$

30. $\dfrac{3x^2}{4} - \dfrac{9y^2}{16} = \dfrac{x^2}{(\quad)} - \dfrac{y^2}{(\quad)}$

31. $\dfrac{x^2}{1/12} - \dfrac{y^2}{2/3} = \dfrac{12x^2}{(\quad)} - \dfrac{3y^2}{(\quad)}$

32. $\dfrac{x^2}{4/9} + \dfrac{y^2}{7/8} = \dfrac{9x^2}{(\quad)} + \dfrac{8y^2}{(\quad)}$

33. $x^{1/3} - 5x^{4/3} = x^{1/3}(\quad)$

34. $3(2x + 1)x^{1/2} + 4x^{3/2} = x^{1/2}(\quad)$

35. $(1 - 3x)^{4/3} - 4x(1 - 3x)^{1/3} = (1 - 3x)^{1/3}(\quad)$

36. $\dfrac{1}{2\sqrt{x}} + 5x^{3/2} - 10x^{5/2} = \dfrac{1}{2\sqrt{x}}(\quad)$

37. $\dfrac{1}{10}(2x + 1)^{5/2} - \dfrac{1}{6}(2x + 1)^{3/2} = \dfrac{(2x + 1)^{3/2}}{15}(\quad)$

38. $\dfrac{3}{7}(t + 1)^{7/3} - \dfrac{3}{4}(t + 1)^{4/3} = \dfrac{3(t + 1)^{4/3}}{28}(\quad)$

In Exercises 39–42, write the expression using negative exponents.

39. $\dfrac{3x^2}{(2x - 1)^3}$

40. $\dfrac{x + 1}{x(6 - x)^{1/2}}$

41. $\dfrac{4}{3x} + \dfrac{4}{x^4} - \dfrac{7x}{\sqrt[3]{2x}}$

42. $\dfrac{x}{x - 2} + \dfrac{1}{x^2} + \dfrac{8}{3(9x)^3}$

In Exercises 43–48, write the fraction as a sum of two or more terms.

43. $\dfrac{16 - 5x - x^2}{x}$

44. $\dfrac{x^3 - 5x^2 + 4}{x^2}$

45. $\dfrac{4x^3 - 7x^2 + 1}{x^{1/3}}$

46. $\dfrac{2x^5 - 3x^3 + 5x - 1}{x^{3/2}}$

47. $\dfrac{3 - 5x^2 - x^4}{\sqrt{x}}$

48. $\dfrac{x^3 - 5x^4}{3x^2}$

In Exercises 49–60, simplify the expression.

49. $\dfrac{-2(x^2 - 3)^{-3}(2x)(x + 1)^3 - 3(x + 1)^2(x^2 - 3)^{-2}}{[(x + 1)^3]^2}$

50. $\dfrac{x^5(-3)(x^2 + 1)^{-4}(2x) - (x^2 + 1)^{-3}(5)x^4}{(x^5)^2}$

51. $\dfrac{(6x + 1)^3(27x^2 + 2) - (9x^3 + 2x)(3)(6x + 1)^2(6)}{[(6x + 1)^3]^2}$

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52. $\dfrac{(4x^2+9)^{1/2}(2)-(2x+3)\left(\frac{1}{2}\right)(4x^2+9)^{-1/2}(8x)}{[(4x^2+9)^{1/2}]^2}$

53. $\dfrac{(x+2)^{3/4}(x+3)^{-2/3}-(x+3)^{1/3}(x+2)^{-1/4}}{[(x+2)^{3/4}]^2}$

54. $(2x-1)^{1/2}-(x+2)(2x-1)^{-1/2}$

55. $\dfrac{2(3x-1)^{1/3}-(2x+1)\left(\frac{1}{3}\right)(3x-1)^{-2/3}(3)}{(3x-1)^{2/3}}$

56. $\dfrac{(x+1)\left(\frac{1}{2}\right)(2x-3x^2)^{-1/2}(2-6x)-(2x-3x^2)^{1/2}}{(x+1)^2}$

57. $\dfrac{1}{(x^2+4)^{1/2}}\cdot\dfrac{1}{2}(x^2+4)^{-1/2}(2x)$

58. $\dfrac{1}{x^2-6}(2x)+\dfrac{1}{2x+5}(2)$

59. $(x^2+5)^{1/2}\left(\frac{3}{2}\right)(3x-2)^{1/2}(3)$
$+(3x-2)^{3/2}\left(\frac{1}{2}\right)(x^2+5)^{-1/2}(2x)$

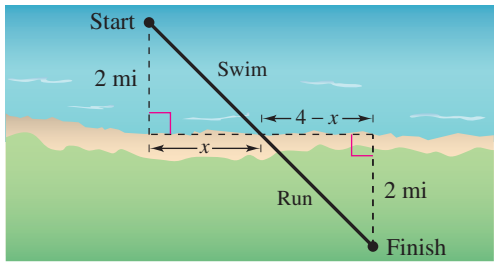
60. $(3x+2)^{-1/2}(3)(x-6)^{1/2}(1)$
$+(x-6)^3\left(-\frac{1}{2}\right)(3x+2)^{-3/2}(3)$

Model It

61. ***Athletics*** An athlete has set up a course for training as part of her regimen in preparation for an upcoming triathlon. She is dropped off by a boat 2 miles from the nearest point on shore. The finish line is 4 miles down the coast and 2 miles inland (see figure). She can swim 2 miles per hour and run 6 miles per hour. The time t (in hours) required for her to reach the finish line can be approximated by the model

$$t=\frac{\sqrt{x^2+4}}{2}+\frac{\sqrt{(4-x)^2+4}}{6}$$

where x is the distance down the coast (in miles) to which she swims and then leaves the water to start her run.



(a) Find the times required for the triathlete to finish when she swims to the points $x=0.5$, $x=1.0,\ldots,x=3.5$, and $x=4.0$ miles down the coast.

(b) Use your results from part (a) to determine the distance down the coast that will yield the minimum amount of time required for the triathlete to reach the finish line.

Model It (continued)

(c) The expression below was obtained using calculus. It can be used to find the minimum amount of time required for the triathlete to reach the finish line. Simplify the expression.

$$\tfrac{1}{2}x(x^2+4)^{-1/2}+\tfrac{1}{6}(x-4)(x^2-8x+20)^{-1/2}$$

62. (a) Verify that $y_1=y_2$ analytically.

$$y_1=x^2\left(\frac{1}{3}\right)(x^2+1)^{-2/3}(2x)+(x^2+1)^{1/3}(2x)$$

$$y_2=\frac{2x(4x^2+3)}{3(x^2+1)^{2/3}}$$

(b) Complete the table and demonstrate the equality in part (a) numerically.

x	-2	-1	$-\frac{1}{2}$	0	1	2	$\frac{5}{2}$
y_1							
y_2							

Synthesis

True or False? **In Exercises 63–66, determine whether the statement is true or false. Justify your answer.**

63. $x^{-1}+y^{-2}=\dfrac{y^2+x}{xy^2}$

64. $\dfrac{1}{x^{-2}+y^{-1}}=x^2+y$

65. $\dfrac{1}{\sqrt{x}+4}=\dfrac{\sqrt{x}-4}{x-16}$

66. $\dfrac{x^2-9}{\sqrt{x}-3}=\sqrt{x}+3$

In Exercises 67–70, find and correct any errors. If the problem is correct, state that it is correct.

67. $x^n\cdot x^{3n}=x^{3n^2}$

68. $(x^n)^{2n}+(x^{2n})^n=2x^{2n^2}$

69. $x^{2n}+y^{2n}=(x^n+y^n)^2$

70. $\dfrac{x^{2n}\cdot x^{3n}}{x^{3n}+x^2}=\dfrac{x^{5n}}{x^{3n}+x^2}$

71. ***Think About It*** You are taking a course in calculus, and for one of the homework problems you obtain the following answer.

$$\tfrac{1}{10}(2x-1)^{5/2}+\tfrac{1}{6}(2x-1)^{3/2}$$

The answer in the back of the book is $\frac{1}{15}(2x-1)^{3/2}(3x+1)$. Show how the second answer can be obtained from the first. Then use the same technique to simplify each of the following expressions.

(a) $\frac{2}{3}x(2x-3)^{3/2}-\frac{2}{15}(2x-3)^{5/2}$

(b) $\frac{2}{3}x(4+x)^{3/2}-\frac{2}{15}(4+x)^{5/2}$

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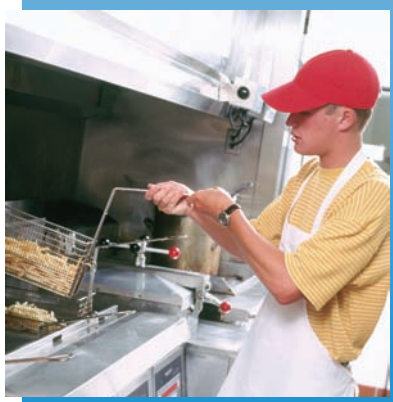
P.7 The Rectangular Coordinate System and Graphs

What you should learn

- Plot points in the Cartesian plane.
- Use the Distance Formula to find the distance between two points.
- Use the Midpoint Formula to find the midpoint of a line segment.
- Use a coordinate plane to model and solve real-life problems.

Why you should learn it

The Cartesian plane can be used to represent relationships between two variables. For instance, in Exercise 61 on page 67, a graph represents the minimum wage in the United States from 1950 to 2004.



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The Cartesian Plane

Just as you can represent real numbers by points on a real number line, you can represent ordered pairs of real numbers by points in a plane called the **rectangular coordinate system,** or the **Cartesian plane,** named after the French mathematician René Descartes (1596–1650).

The Cartesian plane is formed by using two real number lines intersecting at right angles, as shown in Figure P.9. The horizontal real number line is usually called the ***x*-axis,** and the vertical real number line is usually called the ***y*-axis.** The point of intersection of these two axes is the **origin,** and the two axes divide the plane into four parts called **quadrants.**

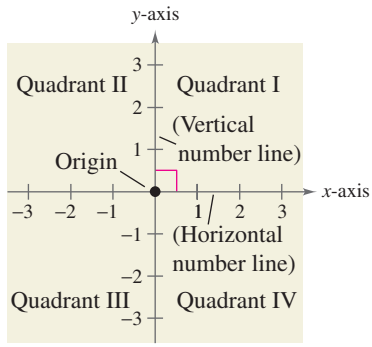


FIGURE P.9

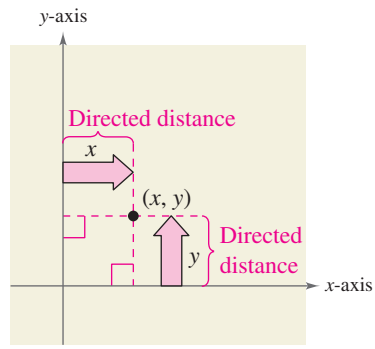


FIGURE P.10

Each point in the plane corresponds to an **ordered pair** (x, y) of real numbers x and y, called **coordinates** of the point. The ***x*-coordinate** represents the directed distance from the y-axis to the point, and the ***y*-coordinate** represents the directed distance from the x-axis to the point, as shown in Figure P.10.



The notation (x, y) denotes both a point in the plane and an open interval on the real number line. The context will tell you which meaning is intended.

Example 1 **Plotting Points in the Cartesian Plane**

Plot the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$.

Solution

To plot the point $(-1, 2)$, imagine a vertical line through -1 on the x-axis and a horizontal line through 2 on the y-axis. The intersection of these two lines is the point $(-1, 2)$. The other four points can be plotted in a similar way, as shown in Figure P.11.

CHECKPOINT Now try Exercise 3.

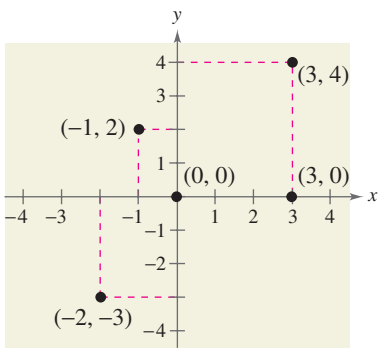


FIGURE P.11

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The beauty of a rectangular coordinate system is that it allows you to *see* relationships between two variables. It would be difficult to overestimate the importance of Descartes's introduction of coordinates *in* the plane. Today, his ideas are in common use in virtually every scientific and business-related field.

Example 2 Sketching a Scatter Plot

From 1990 through 2003, the amounts A (in millions of dollars) spent on skiing equipment in the United States are shown in the table, where t represents the year. Sketch a scatter plot of the data. (Source: National Sporting Goods Association)

Year, t	Amount, A
1990	475
1991	577
1992	521
1993	569
1994	609
1995	562
1996	707
1997	723
1998	718
1999	648
2000	495
2001	476
2002	527
2003	464

Solution

To sketch a *scatter plot* of the data shown in the table, you simply represent each pair of values by an ordered pair (t, A) and plot the resulting points, as shown in Figure P.12. For instance, the first pair of values is represented by the ordered pair $(1990, 475)$. Note that the break in the t-axis indicates that the numbers between 0 and 1990 have been omitted.

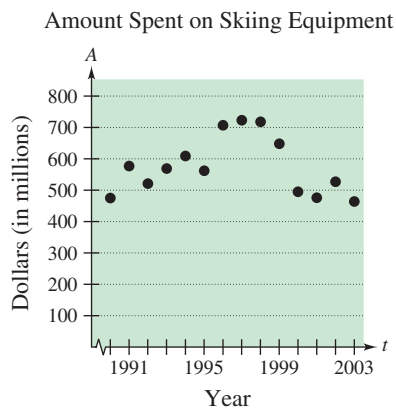


FIGURE P.12

CHECKPOINT Now try Exercise 21.

In Example 2, you could have let $t = 1$ represent the year 1990. In that case, the horizontal axis would not have been broken, and the tick marks would have been labeled 1 through 14 (instead of 1990 through 2003).

Technology

The scatter plot in Example 2 is only one way to represent the data graphically. You could also represent the data using a bar graph and a line graph. If you have access to a graphing utility, try using it to represent graphically the data given in Example 2.

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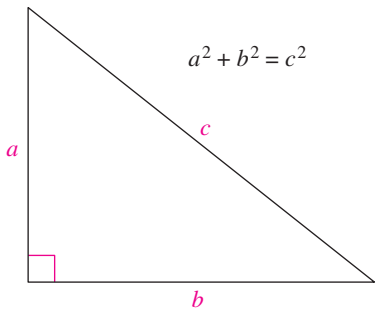


FIGURE P.13

The Distance Formula

Recall from the Pythagorean Theorem that, for a right triangle with hypotenuse of length c and sides of lengths a and b, you have

$$a^2 + b^2 = c^2$$ Pythagorean Theorem

as shown in Figure P.13. (The converse is also true. That is, if $a^2 + b^2 = c^2$, then the triangle is a right triangle.)

Suppose you want to determine the distance d between two points (x_1, y_1) and (x_2, y_2) in the plane. With these two points, a right triangle can be formed, as shown in Figure P.14. The length of the vertical side of the triangle is $|y_2 - y_1|$, and the length of the horizontal side is $|x_2 - x_1|$. By the Pythagorean Theorem, you can write

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This result is the **Distance Formula.**

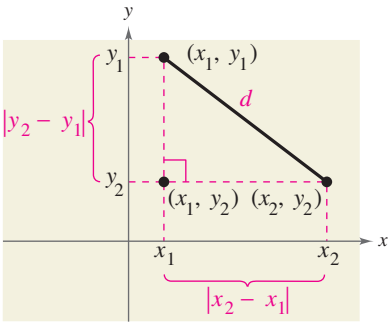


FIGURE P.14

The Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) in the plane is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 3 **Finding a Distance**

Find the distance between the points $(-2, 1)$ and $(3, 4)$.

Algebraic Solution

Let $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, 4)$. Then apply the Distance Formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$ Distance Formula
$$= \sqrt{[3 - (-2)]^2 + (4 - 1)^2}$$ Substitute for x_1, y_1, x_2, and y_2.
$$= \sqrt{(5)^2 + (3)^2}$$ Simplify.
$$= \sqrt{34}$$ Simplify.
$$\approx 5.83$$ Use a calculator.

So, the distance between the points is about 5.83 units. You can use the Pythagorean Theorem to check that the distance is correct.

$$d^2 \stackrel{?}{=} 3^2 + 5^2$$ Pythagorean Theorem
$$\left(\sqrt{34}\right)^2 \stackrel{?}{=} 3^2 + 5^2$$ Substitute for d.
$$34 = 34$$ Distance checks. ✓

CHECKPOINT Now try Exercises 31(a) and (b).

Graphical Solution

Use centimeter graph paper to plot the points $A(-2, 1)$ and $B(3, 4)$. Carefully sketch the line segment from A to B. Then use a centimeter ruler to measure the length of the segment.

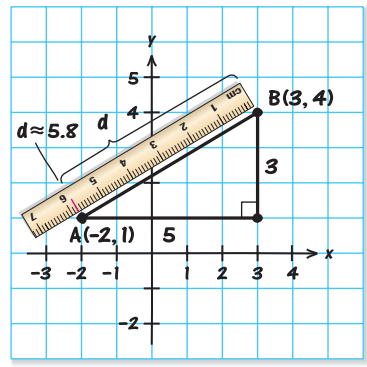


FIGURE P.15

The line segment measures about 5.8 centimeters, as shown in Figure P.15. So, the distance between the points is about 5.8 units.

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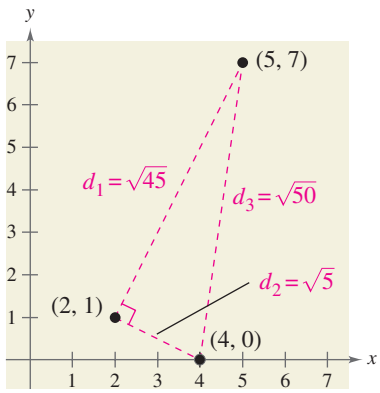


FIGURE P.16

Example 4 Verifying a Right Triangle

Show that the points (2, 1), (4, 0), and (5, 7) are vertices of a right triangle.

Solution

The three points are plotted in Figure P.16. Using the Distance Formula, you can find the lengths of the three sides as follows.

$$d_1 = \sqrt{(5-2)^2 + (7-1)^2} = \sqrt{9+36} = \sqrt{45}$$
$$d_2 = \sqrt{(4-2)^2 + (0-1)^2} = \sqrt{4+1} = \sqrt{5}$$
$$d_3 = \sqrt{(5-4)^2 + (7-0)^2} = \sqrt{1+49} = \sqrt{50}$$

Because

$$(d_1)^2 + (d_2)^2 = 45 + 5 = 50 = (d_3)^2$$

you can conclude by the Pythagorean Theorem that the triangle must be a right triangle.

CHECKPOINT Now try Exercise 41.

The Midpoint Formula

To find the **midpoint** of the line segment that joins two points in a coordinate plane, you can simply find the average values of the respective coordinates of the two endpoints using the **Midpoint Formula.**

The Midpoint Formula

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is given by the Midpoint Formula

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

For a proof of the Midpoint Formula, see Proofs in Mathematics on page 74.

Example 5 Finding a Line Segment's Midpoint

Find the midpoint of the line segment joining the points $(-5, -3)$ and $(9, 3)$.

Solution

Let $(x_1, y_1) = (-5, -3)$ and $(x_2, y_2) = (9, 3)$.

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \qquad \text{Midpoint Formula}$$
$$= \left(\frac{-5 + 9}{2}, \frac{-3 + 3}{2}\right) \qquad \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2.$$
$$= (2, 0) \qquad \text{Simplify.}$$

The midpoint of the line segment is (2, 0), as shown in Figure P.17.

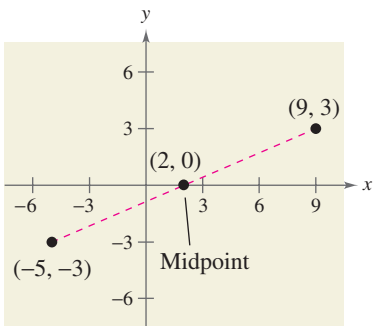


FIGURE P.17

CHECKPOINT Now try Exercise 31(c).

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Applications

Example 6 Finding the Length of a Pass

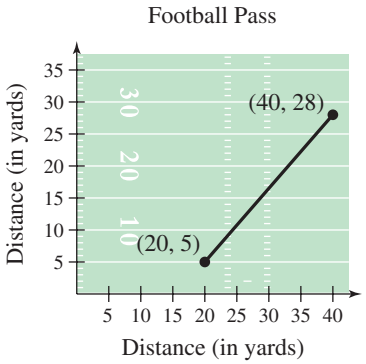


FIGURE P.18

During the third quarter of the 2004 Sugar Bowl, the quarterback for Louisiana State University threw a pass from the 28-yard line, 40 yards from the sideline. The pass was caught by the wide receiver on the 5-yard line, 20 yards from the same sideline, as shown in Figure P.18. How long was the pass?

Solution

You can find the length of the pass by finding the distance between the points (40, 28) and (20, 5).

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance Formula} \\ &= \sqrt{(40 - 20)^2 + (28 - 5)^2} && \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2. \\ &= \sqrt{400 + 529} && \text{Simplify.} \\ &= \sqrt{929} && \text{Simplify.} \\ &\approx 30 && \text{Use a calculator.} \end{aligned}$$

So, the pass was about 30 yards long.

CHECKPOINT Now try Exercise 47.

In Example 6, the scale along the goal line does not normally appear on a football field. However, when you use coordinate geometry to solve real-life problems, you are free to place the coordinate system in any way that is convenient for the solution of the problem.

Example 7 Estimating Annual Revenue

FedEx Corporation had annual revenues of $20.6 billion in 2002 and $24.7 billion in 2004. Without knowing any additional information, what would you estimate the 2003 revenue to have been? (Source: FedEx Corp.)

Solution

One solution to the problem is to assume that revenue followed a linear pattern. With this assumption, you can estimate the 2003 revenue by finding the midpoint of the line segment connecting the points (2002, 20.6) and (2004, 24.7).

$$\begin{aligned} \text{Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) && \text{Midpoint Formula} \\ &= \left(\frac{2002 + 2004}{2}, \frac{20.6 + 24.7}{2}\right) && \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2. \\ &= (2003, 22.65) && \text{Simplify.} \end{aligned}$$

So, you would estimate the 2003 revenue to have been about $22.65 billion, as shown in Figure P.19. (The actual 2003 revenue was $22.5 billion.)

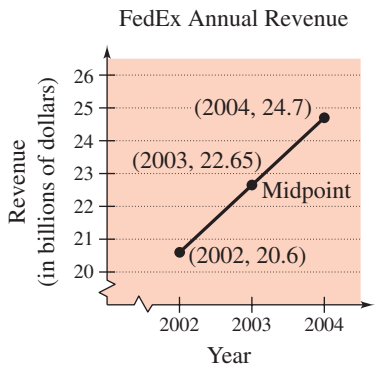


FIGURE P.19

CHECKPOINT Now try Exercise 49.

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Paul Morrell

Much of computer graphics, including this computer-generated goldfish tessellation, consists of transformations of points in a coordinate plane. One type of transformation, a translation, is illustrated in Example 8. Other types include reflections, rotations, and stretches.

Example 8 **Translating Points in the Plane**

The triangle in Figure P.20 has vertices at the points $(-1, 2)$, $(1, -4)$, and $(2, 3)$. Shift the triangle three units to the right and two units upward and find the vertices of the shifted triangle, as shown in Figure P.21.

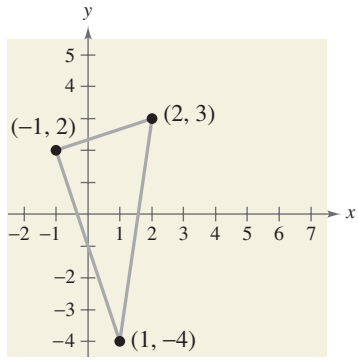


FIGURE P.20

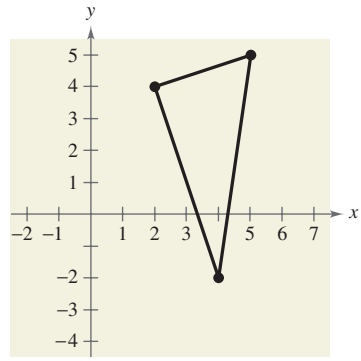


FIGURE P.21

Solution

To shift the vertices three units to the right, add 3 to each of the x-coordinates. To shift the vertices two units upward, add 2 to each of the y-coordinates.

Original Point	*Translated Point*
$(-1, 2)$	$(-1 + 3, 2 + 2) = (2, 4)$
$(1, -4)$	$(1 + 3, -4 + 2) = (4, -2)$
$(2, 3)$	$(2 + 3, 3 + 2) = (5, 5)$

CHECKPOINT Now try Exercise 51.

The figures provided with Example 8 were not really essential to the solution. Nevertheless, it is strongly recommended that you develop the habit of including sketches with your solutions—even if they are not required.

WRITING ABOUT MATHEMATICS

Extending the Example Example 8 shows how to translate points in a coordinate plane. Write a short paragraph describing how each of the following transformed points is related to the original point.

Original Point	*Transformed Point*
(x, y)	$(-x, y)$
(x, y)	$(x, -y)$
(x, y)	$(-x, -y)$

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P.7 Exercises

VOCABULARY CHECK

1. Match each term with its definition.

(a) x-axis	(i) point of intersection of vertical axis and horizontal axis
(b) y-axis	(ii) directed distance from the x-axis
(c) origin	(iii) directed distance from the y-axis
(d) quadrants	(iv) four regions of the coordinate plane
(e) x-coordinate	(v) horizontal real number line
(f) y-coordinate	(vi) vertical real number line

In Exercises 2–4, fill in the blanks.

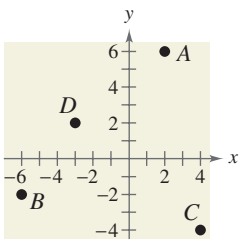
2. An ordered pair of real numbers can be represented in a plane called the rectangular coordinate system or the _____ plane.

3. The _____ _____ is a result derived from the Pythagorean Theorem.

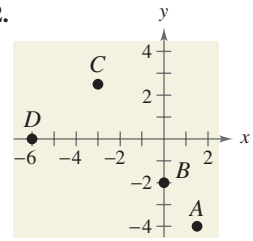
4. Finding the average values of the representative coordinates of the two endpoints of a line segment in a coordinate plane is also known as using the _____ _____.

In Exercises 1 and 2, approximate the coordinates of the points.

1.



2.



In Exercises 3–6, plot the points in the Cartesian plane.

3. $(-4, 2), (-3, -6), (0, 5), (1, -4)$

4. $(0, 0), (3, 1), (-2, 4), (1, -1)$

5. $(3, 8), (0.5, -1), (5, -6), (-2, 2.5)$

6. $\left(1, -\frac{1}{3}\right), \left(\frac{3}{4}, 3\right), (-3, 4), \left(-\frac{4}{3}, -\frac{3}{2}\right)$

In Exercises 7–10, find the coordinates of the point.

7. The point is located three units to the left of the y-axis and four units above the x-axis.

8. The point is located eight units below the x-axis and four units to the right of the y-axis.

9. The point is located five units below the x-axis and the coordinates of the point are equal.

10. The point is on the x-axis and 12 units to the left of the y-axis.

In Exercises 11–20, determine the quadrant(s) in which (x, y) is located so that the condition(s) is (are) satisfied.

11. $x > 0$ and $y < 0$

12. $x < 0$ and $y < 0$

13. $x = -4$ and $y > 0$

14. $x > 2$ and $y = 3$

15. $y < -5$

16. $x > 4$

17. $x < 0$ and $-y > 0$

18. $-x > 0$ and $y < 0$

19. $xy > 0$

20. $xy < 0$

In Exercises 21 and 22, sketch a scatter plot of the data shown in the table.

21. ***Number of Stores*** The table shows the number y of Wal-Mart stores for each year x from 1996 through 2003. (Source: Wal-Mart Stores, Inc.)

Year, x	Number of stores, y
1996	3054
1997	3406
1998	3599
1999	3985
2000	4189
2001	4414
2002	4688
2003	4906

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22. ***Meteorology*** The table shows the lowest temperature on record y (in degrees Fahrenheit) in Duluth, Minnesota, for each month x, where $x = 1$ represents January. (Source: NOAA)

Month, x	Temperature, y
1	−39
2	−39
3	−29
4	−5
5	17
6	27
7	35
8	32
9	22
10	8
11	−23
12	−34

In Exercises 23–26, find the distance between the points. (*Note:* In each case, the two points lie on the same horizontal or vertical line.)

23. $(6, -3), (6, 5)$

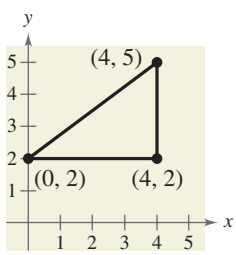
24. $(1, 4), (8, 4)$

25. $(-3, -1), (2, -1)$

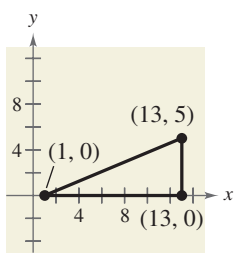
26. $(-3, -4), (-3, 6)$

In Exercises 27–30, (a) find the length of each side of the right triangle, and (b) show that these lengths satisfy the Pythagorean Theorem.

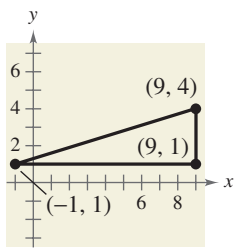
27.



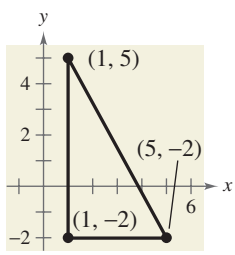
28.



29.



30.



In Exercises 31–40, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

31. $(1, 1), (9, 7)$

32. $(1, 12), (6, 0)$

33. $(-4, 10), (4, -5)$

34. $(-7, -4), (2, 8)$

35. $(-1, 2), (5, 4)$

36. $(2, 10), (10, 2)$

37. $\left(\frac{1}{2}, 1\right), \left(-\frac{5}{2}, \frac{4}{3}\right)$

38. $\left(-\frac{1}{3}, -\frac{1}{3}\right), \left(-\frac{1}{6}, -\frac{1}{2}\right)$

39. $(6.2, 5.4), (-3.7, 1.8)$

40. $(-16.8, 12.3), (5.6, 4.9)$

In Exercises 41 and 42, show that the points form the vertices of the indicated polygon.

41. Right triangle: $(4, 0), (2, 1), (-1, -5)$

42. Isosceles triangle: $(1, -3), (3, 2), (-2, 4)$

43. A line segment has (x_1, y_1) as one endpoint and (x_m, y_m) as its midpoint. Find the other endpoint (x_2, y_2) of the line segment in terms of x_1, y_1, x_m, and y_m.

44. Use the result of Exercise 43 to find the coordinates of the endpoint of a line segment if the coordinates of the other endpoint and midpoint are, respectively,

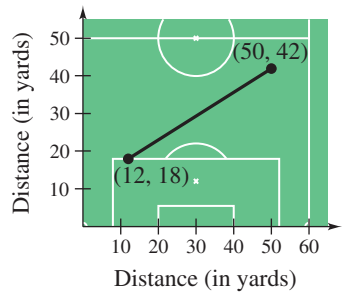
(a) $(1, -2), (4, -1)$ and (b) $(-5, 11), (2, 4)$.

45. Use the Midpoint Formula three times to find the three points that divide the line segment joining (x_1, y_1) and (x_2, y_2) into four parts.

46. Use the result of Exercise 45 to find the points that divide the line segment joining the given points into four equal parts.

(a) $(1, -2), (4, -1)$ (b) $(-2, -3), (0, 0)$

47. ***Sports*** A soccer player passes the ball from a point that is 18 yards from the endline and 12 yards from the sideline. The pass is received by a teammate who is 42 yards from the same endline and 50 yards from the same sideline, as shown in the figure. How long is the pass?



48. ***Flying Distance*** An airplane flies from Naples, Italy in a straight line to Rome, Italy, which is 120 kilometers north and 150 kilometers west of Naples. How far does the plane fly?

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Sales **In Exercises 49 and 50, use the Midpoint Formula to estimate the sales of Big Lots, Inc. and Dollar Tree Stores, Inc. in 2002, given the sales in 2001 and 2003. Assume that the sales followed a linear pattern.** (Source: Big Lots, Inc.; Dollar Tree Stores, Inc.)

49. Big Lots

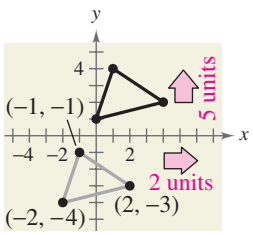
Year	Sales (in millions)
2001	$3433
2003	$4174

50. Dollar Tree

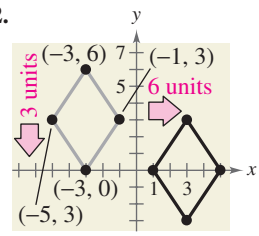
Year	Sales (in millions)
2001	$1987
2003	$2800

In Exercises 51–54, the polygon is shifted to a new position in the plane. Find the coordinates of the vertices of the polygon in its new position.

51.



52.



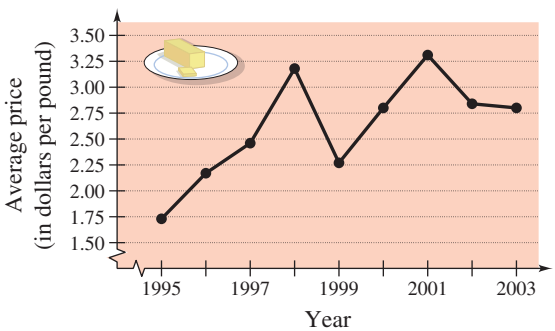
53. Original coordinates of vertices: $(-7, -2), (-2, 2), (-2, -4), (-7, -4)$

Shift: eight units upward, four units to the right

54. Original coordinates of vertices: (5, 8), (3, 6), (7, 6), (5, 2)

Shift: 6 units downward, 10 units to the left

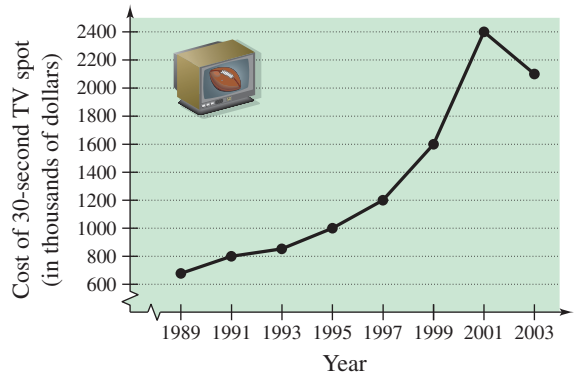
Retail Price **In Exercises 55 and 56, use the graph below, which shows the average retail price of 1 pound of butter from 1995 to 2003.** (Source: U.S. Bureau of Labor Statistics)



55. Approximate the highest price of a pound of butter shown in the graph. When did this occur?

56. Approximate the percent change in the price of butter from the price in 1995 to the highest price shown in the graph.

Advertising **In Exercises 57 and 58, use the graph below, which shows the cost of a 30-second television spot (in thousands of dollars) during the Super Bowl from 1989 to 2003.** (Source: USA Today Research and CNN)



57. Approximate the percent increase in the cost of a 30-second spot from Super Bowl XXIII in 1989 to Super Bowl XXXV in 2001.

58. Estimate the percent increase in the cost of a 30-second spot (a) from Super Bowl XXIII in 1989 to Super Bowl XXVII in 1993 and (b) from Super Bowl XXVII in 1993 to Super Bowl XXXVII in 2003.

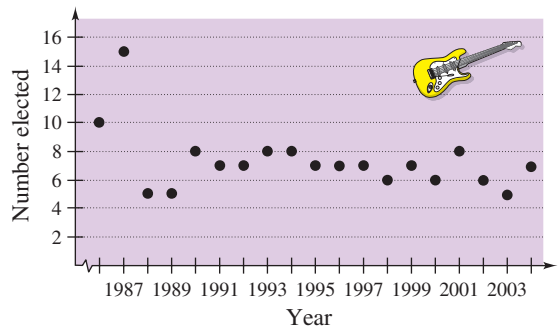
59. ***Make a Conjecture*** Plot the points $(2, 1), (-3, 5)$, and $(7, -3)$ on a rectangular coordinate system. Then change the sign of the x-coordinate of each point and plot the three new points on the same rectangular coordinate system. Make a conjecture about the location of a point when each of the following occurs.

(a) The sign of the x-coordinate is changed.

(b) The sign of the y-coordinate is changed.

(c) The signs of both the x- and y-coordinates are changed.

60. ***Music*** The graph shows the numbers of recording artists who were elected to the Rock and Roll Hall of Fame from 1986 to 2004.



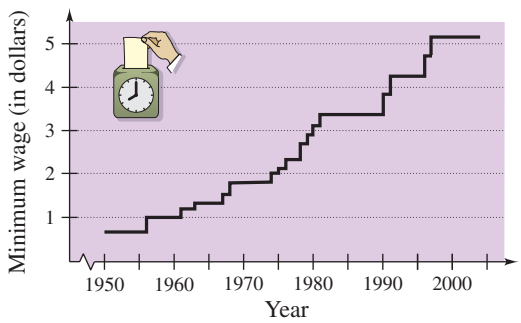
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(a) Describe any trends in the data. From these trends, predict the number of artists elected in 2008.

(b) Why do you think the numbers elected in 1986 and 1987 were greater than in other years?

Model It

61. ***Labor Force*** Use the graph below, which shows the minimum wage in the United States (in dollars) from 1950 to 2004. (Source: U.S. Department of Labor)



(a) Which decade shows the greatest increase in minimum wage?

(b) Approximate the percent increases in the minimum wage from 1990 to 1995 and from 1995 to 2004.

(c) Use the percent increase from 1995 to 2004 to predict the minimum wage in 2008.

(d) Do you believe that your prediction in part (c) is reasonable? Explain.

62. ***Sales*** Pepsi Bottling Group, Inc. had sales of $6603 million in 1996 and $10,800 million in 2004. Use the Midpoint Formula to estimate the sales in 1998, 2000, and 2002. Assume that the sales followed a linear pattern. (Source: Pepsi Bottling Group, Inc.)

63. ***Sales*** The Coca-Cola Company had sales of $18,546 million in 1996 and $21,900 million in 2004. Use the Midpoint Formula to estimate the sales in 1998, 2000, and 2002. Assume that the sales followed a linear pattern. (Source: The Coca-Cola Company)

64. ***Data Analysis: Exam Scores*** The table shows the mathematics entrance test scores x and the final examination scores y in an algebra course for a sample of 10 students.

x	22	29	35	40	44
y	53	74	57	66	79

x	48	53	58	65	76
y	90	76	93	83	99

(a) Sketch a scatter plot of the data.

(b) Find the entrance exam score of any student with a final exam score in the 80s.

(c) Does a higher entrance exam score imply a higher final exam score? Explain.

65. ***Data Analysis: Mail*** The table shows the number y of pieces of mail handled (in billions) by the U.S. Postal Service for each year x from 1996 through 2003. (Source U.S. Postal Service)

Year, x	Pieces of mail, y
1996	183
1997	191
1998	197
1999	202
2000	208
2001	207
2002	203
2003	202

(a) Sketch a scatter plot of the data.

(b) Approximate the year in which there was the greatest decrease in the number of pieces of mail handled.

(c) Why do you think the number of pieces of mail handled decreased?

66. ***Data Analysis: Oil Production*** The table shows the daily crude oil production y (in thousands of barrels) in Saudi Arabia for each year x from 1995 through 2001. (Source: U.S. Energy Information Administration)

Year, x	Oil production, y
1995	8231
1996	8218
1997	8362
1998	8389
1999	7833
2000	8404
2001	8031

(a) Sketch a scatter plot of the data.

(b) In which year was the daily oil production the greatest? In which year was it the least?

(c) Determine the percent increase in daily oil production between the years you found in part (b).

(d) Use the scatter plot to describe any trends in the data.

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67. ***Data Analysis: Athletics*** The table shows the numbers of men's M and women's W college basketball teams for each year x from 1994 through 2003. (Source: National Collegiate Athletic Association)

Year, x	Men's teams, M	Women's teams, W
1994	858	859
1995	868	864
1996	866	874
1997	865	879
1998	895	911
1999	926	940
2000	932	956
2001	937	958
2002	936	975
2003	967	1009

(a) Sketch a scatter plot of both sets of data on the same set of coordinate axes.

(b) Find the year in which the numbers of men's and women's teams were nearly equal.

(c) Find the year in which the difference between the numbers of men's and women's teams was the greatest. What is this difference?

68. ***Collinear Points*** Three or more points are *collinear* if they all lie on the same line. Use the steps below to determine if the set of points $\{A(2, 3), B(2, 6), C(6, 3)\}$ and the set of points $\{A(8, 3), B(5, 2), C(2, 1)\}$ are collinear.

(a) For each set of points, use the Distance Formula to find the distances from A to B, from B to C, and from A to C. What relationship exists among these distances for each set of points?

(b) Plot each set of points in the Cartesian plane. Do all the points of either set appear to lie on the same line?

(c) Compare your conclusions from part (a) with the conclusions you made from the graphs in part (b). Make a general statement about how to use the Distance Formula to determine collinearity.

Synthesis

True or False? **In Exercises 69 and 70, determine whether the statement is true or false. Justify your answer.**

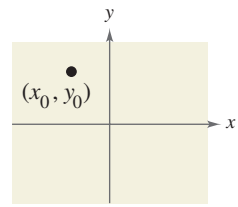
69. In order to divide a line segment into 16 equal parts, you would have to use the Midpoint Formula 16 times.

70. The points $(-8, 4)$, $(2, 11)$, and $(-5, 1)$ represent the vertices of an isosceles triangle.

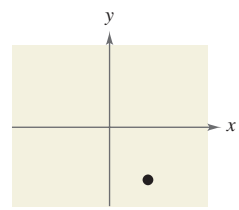
71. ***Think About It*** What is the y-coordinate of any point on the x-axis? What is the x-coordinate of any point on the y-axis?

72. ***Think About It*** When plotting points on the rectangular coordinate system, is it true that the scales on the x- and y-axes must be the same? Explain.

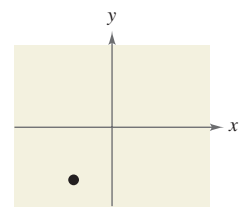
In Exercises 73–76, use the plot of the point (x_0, y_0) in the figure. Match the transformation of the point with the correct plot. [The plots are labeled (a), (b), (c), and (d).]



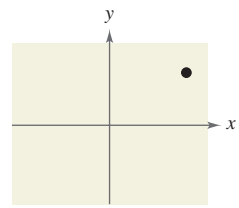
(a)



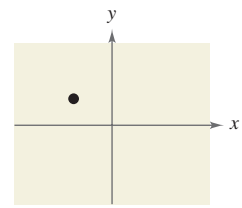
(b)



(c)



(d)



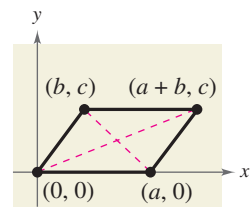
73. $(x_0, -y_0)$

74. $(-2x_0, y_0)$

75. $\left(x_0, \frac{1}{2}y_0\right)$

76. $(-x_0, -y_0)$

77. ***Proof*** Prove that the diagonals of the parallelogram in the figure intersect at their midpoints.



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P Chapter Summary

What did you learn?

Section P.1	Review Exercises
☐ Represent and classify real numbers *(p. 2)*.	1, 2
☐ Order real numbers and use inequalities *(p. 3)*.	3–6
☐ Find the absolute values of real numbers and find the distance between two real numbers *(p. 5)*.	7–12
☐ Evaluate algebraic expressions *(p. 6)*.	13–16
☐ Use the basic rules and properties of algebra *(p. 8)*.	17–26
Section P.2	
☐ Use properties of exponents *(p. 12)*.	27–30
☐ Use scientific notation to represent real numbers *(p. 14)*.	31–34
☐ Use properties of radicals *(p. 15)* to simplify and combine radicals *(p. 17)*.	35–42
☐ Rationalize denominators and numerators *(p. 18)*.	43–46
☐ Use properties of rational exponents *(p. 19)*.	47–50
Section P.3	
☐ Write polynomials in standard form *(p. 24)*.	51–54
☐ Add, subtract, and multiply polynomials *(p. 25)*.	55–64
☐ Use polynomials to solve real-life problems *(p. 28)*.	65, 66
Section P.4	
☐ Remove common factors from polynomials *(p. 33)*.	67, 68
☐ Factor special polynomial forms *(p. 34)*.	69–72
☐ Factor trinomials as the product of two binomials *(p. 36)*.	73, 74
☐ Factor polynomials by grouping *(p. 37)*.	75, 76
Section P.5	
☐ Find domains of algebraic expressions *(p. 41)*.	77, 78
☐ Simplify rational expressions *(p. 42)*.	79, 80
☐ Add, subtract, multiply, and divide rational expressions *(p. 43)*.	81–84
☐ Simplify complex fractions *(p. 45)*.	85–88
Section P.6	
☐ Avoid common algebraic errors *(p. 51)*.	89–94
☐ Recognize and use algebraic techniques that are common in calculus *(p. 53)*.	95–102
Section P.7	
☐ Plot points in the Cartesian plane *(p. 58)*.	103–106
☐ Use the Distance Formula to find the distance between two points *(p. 60)*.	107–110
☐ Use the Midpoint Formula to find the midpoint of a line segment *(p. 61)*.	107–110
☐ Use a coordinate plane to model and solve real-life problems *(p. 62)*.	111–114

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P Review Exercises

P.1 **In Exercises 1 and 2, determine which numbers in the set are (a) natural numbers, (b) whole numbers, (c) integers, (d) rational numbers, and (e) irrational numbers.**

1. $\{11, -14, -\frac{8}{9}, \frac{5}{2}, \sqrt{6}, 0.4\}$
2. $\{\sqrt{15}, -22, -\frac{10}{3}, 0, 5.2, \frac{3}{7}\}$

In Exercises 3 and 4, use a calculator to find the decimal form of each rational number. If it is a nonterminating decimal, write the repeating pattern. Then plot the numbers on the real number line and place the appropriate inequality sign (< or >) between them.

3. (a) $\frac{5}{6}$ (b) $\frac{7}{8}$
4. (a) $\frac{9}{25}$ (b) $\frac{5}{7}$

In Exercises 5 and 6, give a verbal description of the subset of real numbers represented by the inequality, and sketch the subset on the real number line.

5. $x \leq 7$
6. $x > 1$

In Exercises 7 and 8, find the distance between a and b.

7. $a = -92, b = 63$
8. $a = -112, b = -6$

In Exercises 9–12, use absolute value notation to describe the situation.

9. The distance between x and 7 is at least 4.
10. The distance between x and 25 is no more than 10.
11. The distance between y and -30 is less than 5.
12. The distance between z and -16 is greater than 8.

In Exercises 13–16, evaluate the expression for each value of x. (If not possible, state the reason.)

	Expression	*Values*	
13.	$12x - 7$	(a) $x = 0$	(b) $x = -1$
14.	$x^2 - 6x + 5$	(a) $x = -2$	(b) $x = 2$
15.	$-x^2 + x - 1$	(a) $x = 1$	(b) $x = -1$
16.	$\frac{x}{x-3}$	(a) $x = -3$	(b) $x = 3$

In Exercises 17–20, identify the rule of algebra illustrated by the statement.

17. $2x + (3x - 10) = (2x + 3x) - 10$
18. $4(t + 2) = 4 \cdot t + 4 \cdot 2$
19. $0 + (a - 5) = a - 5$
20. $\frac{2}{y+4} \cdot \frac{y+4}{2} = 1, \quad y \neq -4$

In Exercises 21–26, perform the operation without using a calculator.

21. $|-3| + 4(-2) - 6$
22. $\frac{|-10|}{-10}$
23. $\frac{5}{18} \div \frac{10}{3}$
24. $(16 - 8) \div 4$
25. $6[4 - 2(6 + 8)]$
26. $-4[16 - 3(7 - 10)]$

P.2 **In Exercises 27 and 28, simplify each expression.**

27. (a) $3x^2(4x^3)^3$ (b) $\frac{5y^6}{10y}$
28. (a) $(-2z)^2(8z^3)$ (b) $\frac{36x^5}{9x^{10}}$

In Exercises 29 and 30, rewrite each expression with positive exponents and simplify.

29. (a) $\frac{6^2u^3v^{-3}}{12u^{-2}v}$ (b) $\frac{3^{-4}m^{-1}n^{-3}}{9^{-2}mn^{-3}}$
30. (a) $(x + y^{-1})^{-1}$ (b) $\left(\frac{x^{-3}}{y}\right)\left(\frac{x}{y}\right)^{-1}$

In Exercises 31 and 32, write the number in scientific notation.

31. *Sales for Pep Boys in 2003:* \$2,134,300,000 (Source: Pep Boys)
32. *Number of meters in 1 foot:* 0.3048

In Exercises 33 and 34, write the number in decimal notation.

33. *Distance between the sun and Jupiter:* 4.837×10^8 miles
34. *Ratio of day to year:* 2.74×10^{-3}

In Exercises 35–38, simplify each expression.

35. (a) $\sqrt[3]{27^2}$ (b) $\sqrt{49^3}$
36. (a) $\sqrt[3]{\frac{64}{125}}$ (b) $\sqrt{\frac{81}{100}}$
37. (a) $\left(\sqrt[3]{216}\right)^3$ (b) $\sqrt[4]{32^4}$
38. (a) $\sqrt[3]{\frac{2x^3}{27}}$ (b) $\sqrt[5]{64x^6}$

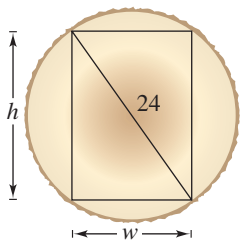
In Exercises 39 and 40, simplify each expression.

39. (a) $\sqrt{50} - \sqrt{18}$ (b) $2\sqrt{32} + 3\sqrt{72}$
40. (a) $\sqrt{8x^3} + \sqrt{2x}$ (b) $\sqrt{18x^5} - \sqrt{8x^3}$

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41. ***Writing*** Explain why $\sqrt{5u} + \sqrt{3u} \neq 2\sqrt{2u}$.

42. ***Engineering*** The rectangular cross section of a wooden beam cut from a log of diameter 24 inches (see figure) will have a maximum strength if its width w and height h are $w = 8\sqrt{3}$ and $h = \sqrt{24^2 - (8\sqrt{3})^2}$. Find the area of the rectangular cross section and write the answer in simplest form.



In Exercises 43 and 44, rationalize the denominator of the expression. Then, simplify your answer.

43. $\dfrac{1}{2 - \sqrt{3}}$

44. $\dfrac{1}{\sqrt{5} + 1}$

In Exercises 45 and 46, rationalize the numerator of the expression. Then, simplify your answer.

45. $\dfrac{\sqrt{7} + 1}{2}$

46. $\dfrac{\sqrt{2} - \sqrt{11}}{3}$

In Exercises 47–50, simplify the expression.

47. $(16)^{3/2}$

48. $(64)^{-2/3}$

49. $(3x^{2/5})(2x^{1/2})$

50. $(x - 1)^{1/3}(x - 1)^{-1/4}$

P.3 In Exercises 51–54, write the polynomial in standard form. Identify the degree and leading coefficient.

51. $3 - 11x^2$

52. $3x^3 - 5x^5 + x - 4$

53. $-4 - 12x^2$

54. $12x - 7x^2 + 6$

In Exercises 55–58, perform the operation and write the result in standard form.

55. $-(3x^2 + 2x) + (1 - 5x)$

56. $8y - [2y^2 - (3y - 8)]$

57. $2x(x^2 - 5x + 6)$

58. $(3x^3 - 1.5x^2 + 4)(-3x)$

In Exercises 59–64, find the product.

59. $(3x - 6)(5x + 1)$

60. $\left(x - \dfrac{1}{x}\right)(x + 2)$

61. $(2x - 3)^2$

62. $(6x + 5)(6x - 5)$

63. $\left(3\sqrt{5} + 2\right)\left(3\sqrt{5} - 2\right)$

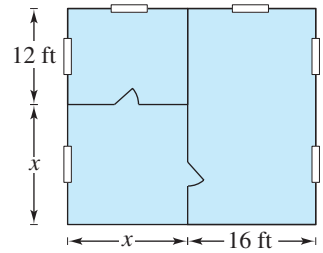
64. $(x - 4)^3$

65. ***Surface Area*** The surface area S of a right circular cylinder is $S = 2\pi r^2 + 2\pi rh$.

(a) Draw a right circular cylinder of radius r and height h. Use the figure to explain how the surface area formula was obtained.

(b) Find the surface area when the radius is 6 inches and the height is 8 inches.

66. ***Geometry*** Find a polynomial that represents the total number of square feet for the floor plan shown in the figure.



P.4 In Exercises 67–76, completely factor the expression.

67. $x^3 - x$

68. $x(x - 3) + 4(x - 3)$

69. $25x^2 - 49$

70. $x^2 - 12x + 36$

71. $x^3 - 64$

72. $8x^3 + 27$

73. $2x^2 + 21x + 10$

74. $3x^2 + 14x + 8$

75. $x^3 - x^2 + 2x - 2$

76. $x^3 - 4x^2 + 2x - 8$

P.5 In Exercises 77 and 78, find the domain of the expression.

77. $\dfrac{1}{x + 6}$

78. $\sqrt{x + 4}$

In Exercises 79 and 80, write the rational expression in simplest form.

79. $\dfrac{x^2 - 64}{5(3x + 24)}$

80. $\dfrac{x^3 + 27}{x^2 + x - 6}$

In Exercises 81–84, perform the indicated operation and simplify.

81. $\dfrac{x^2 - 4}{x^4 - 2x^2 - 8} \cdot \dfrac{x^2 + 2}{x^2}$

82. $\dfrac{4x - 6}{(x - 1)^2} \div \dfrac{2x^2 - 3x}{x^2 + 2x - 3}$

83. $\dfrac{1}{x - 1} + \dfrac{1 - x}{x^2 + x + 1}$

84. $\dfrac{3x}{x + 2} - \dfrac{4x^2 - 5}{2x^2 + 3x - 2}$

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In Exercises 85 and 86, simplify the complex fraction.

85. $\dfrac{\left[\dfrac{3a}{(a^2/x) - 1}\right]}{\left(\dfrac{a}{x} - 1\right)}$

86. $\dfrac{\left(\dfrac{1}{2x - 3} - \dfrac{1}{2x + 3}\right)}{\left(\dfrac{1}{2x} - \dfrac{1}{2x + 3}\right)}$

In Exercises 87 and 88, simplify the difference quotient.

87. $\dfrac{\left[\dfrac{1}{2(x + h)} - \dfrac{1}{2x}\right]}{h}$

88. $\dfrac{\dfrac{1}{x + h - 3} - \dfrac{1}{x - 3}}{h}$

P.6 In Exercises 89–94, describe and correct the error.

89. $10(4 \cdot 7) = 40 \cdot 70$

90. $\left(\frac{1}{3}x\right)\left(\frac{1}{3}y\right) = \frac{1}{3}xy$

91. $(3^4)^4 = 3^8$

92. $\sqrt{3^2 + 4^2} = 3 + 4$

93. $(5 + 8)^2 = 5^2 + 8^2$

94. $\dfrac{7 + 5(x + 3)}{x + 3} = 12$

In Exercises 95–98, insert the required factor in the parentheses.

95. $\frac{2}{3}x^4 - \frac{3}{8}x^3 + \frac{5}{6}x^2 = \frac{1}{24}x^2(\quad)$

96. $\dfrac{t}{\sqrt{t + 1}} - \sqrt{t + 1} = \dfrac{1}{\sqrt{t + 1}}(\quad)$

97. $2x(x^2 - 3)^{1/3} - 5(x^2 - 3)^{4/3} = (x^2 - 3)^{1/3}(\quad)$

98. $y(y - 1)^{5/4} - y^2(y - 1)^{1/4} = y(y - 1)^{1/4}(\quad)$

In Exercises 99 and 100, write the fraction as a sum of two or more terms.

99. $\dfrac{x^3 + 5x^2 + 7}{x}$

100. $\dfrac{2x^3 - x^2 + 4}{x^{1/2}}$

In Exercises 101 and 102, simplify the expression.

101. $\dfrac{x(x + 2)^{-1/2} + (x + 2)^{1/2}}{(x + 2)^{3/2}}$

102. $\dfrac{\frac{2}{3}x(4 + x)^{-1/2} - \frac{2}{15}(4 + x)^{1/2}}{(4 + x)^{5/2}}$

P.7 In Exercises 103 and 104, plot the points in the Cartesian plane.

103. $(2, 2), (0, -4), (-3, 6), (-1, -7)$

104. $(5, 0), (8, 1), (4, -2), (-3, -3)$

In Exercises 105 and 106, determine the quadrant(s) in which (x, y) is located so that the condition(s) is (are) satisfied.

105. $x > 0$ and $y = -2$

106. $y > 0$

In Exercises 107–110, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

107. $(-3, 8), (1, 5)$

108. $(-2, 6), (4, -3)$

109. $(5.6, 0), (0, 8.2)$

110. $(0, -1.2), (-3.6, 0)$

In Exercises 111 and 112, the polygon is shifted to a new position in the plane. Find the coordinates of the vertices of the polygon in its new position.

111. Original coordinates of vertices:

$(4, 8), (6, 8), (4, 3), (6, 3)$

Shift: three units downward, two units to the left

112. Original coordinates of vertices:

$(0, 1), (3, 3), (0, 5), (-3, 3)$

Shift: five units upward, four units to the left

113. ***Sales*** The Cheesecake Factory had annual sales of \$539.1 million in 2001 and \$773.8 million in 2003. Use the Midpoint Formula to estimate the sales in 2002. (Source: The Cheesecake Factory, Inc.)

114. ***Meteorology*** The apparent temperature is a measure of relative discomfort to a person from heat and high humidity. The table shows the actual temperatures x (in degrees Fahrenheit) versus the apparent temperatures y (in degrees Fahrenheit) for a relative humidity of 75%.

x	70	75	80	85	90	95	100
y	70	77	85	95	109	130	150

(a) Sketch a scatter plot of the data shown in the table.

(b) Find the change in the apparent temperature when the actual temperature changes from 70°F to 100°F.

Synthesis

True or False? **In Exercises 115 and 116, determine whether the statement is true or false. Justify your answer.**

115. A binomial sum squared is equal to the sum of the terms squared.

116. $x^n - y^n$ factors as conjugates for all values of n.

117. ***Think About It*** Is the following statement true for all nonzero real numbers a and b? Explain.

$$\frac{ax - b}{b - ax} = -1$$

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P Chapter Test

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

1. Place $<$ or $>$ between the real numbers $-\frac{10}{3}$ and $-|-4|$.

2. Find the distance between the real numbers -5.4 and $3\frac{3}{4}$.

3. Identify the rule of algebra illustrated by $(5 - x) + 0 = 5 - x$.

In Exercises 4 and 5, evaluate each expression without using a calculator.

4. (a) $27\left(-\frac{2}{3}\right)$ (b) $\frac{5}{18} \div \frac{5}{8}$ (c) $\left(-\frac{3}{5}\right)^3$ (d) $\left(\frac{3^2}{2}\right)^{-3}$

5. (a) $\sqrt{5} \cdot \sqrt{125}$ (b) $\frac{\sqrt{27}}{\sqrt{2}}$ (c) $\frac{5.4 \times 10^8}{3 \times 10^3}$ (d) $(3 \times 10^4)^3$

In Exercises 6 and 7, simplify each expression.

6. (a) $3z^2(2z^3)^2$ (b) $(u - 2)^{-4}(u - 2)^{-3}$ (c) $\left(\frac{x^{-2}y^2}{3}\right)^{-1}$

7. (a) $9z\sqrt{8z} - 3\sqrt{2z^3}$ (b) $(4x^{3/5})(x^{1/3})$ (c) $\sqrt[3]{\frac{16}{y^5}}$

8. Write the polynomial $x + 4x^4 - 5 - 3x^2$ in standard form. Identify the degree and leading coefficient.

In Exercises 9–12, perform the operation and simplify.

9. $(x^2 + 3) - [3x + (8 - x^2)]$

10. $(x + \sqrt{5})(x - \sqrt{5})$

11. $\frac{8x}{x - 3} + \frac{24}{3 - x}$

12. $\frac{\left(\frac{2}{x} - \frac{2}{x + 1}\right)}{\left(\frac{4}{x^2 - 1}\right)}$

13. Factor (a) $2x^4 - 3x^3 - 2x^2$ and (b) $x^3 + 2x^2 - 4x - 8$ completely.

14. Rationalize each denominator. (a) $\frac{16}{\sqrt[3]{16}}$ (b) $\frac{6}{1 - \sqrt{3}}$

15. Find the domain of $\frac{1 - x}{4 - x}$.

16. Multiply: $\frac{y^2 + 8y + 16}{2y - 4} \cdot \frac{8y - 16}{(y + 4)^3}$.

17. A T-shirt company can produce and sell x T-shirts per day. The total cost C (in dollars) for producing x T-shirts is $C = 1480 + 6x$, and the total revenue R (in dollars) is $R = 15x$. Find the profit obtained by selling 225 T-shirts per day.

18. Plot the points $(-2, 5)$ and $(6, 0)$. Find the coordinates of the midpoint of the line segment joining the points and the distance between the points.

19. Write an expression for the area of the shaded region in the figure at the left, and simplify the result.

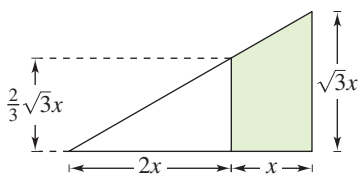


FIGURE FOR 19

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Proofs in Mathematics

What does the word *proof* mean to you? In mathematics, the word *proof* is used to mean simply a valid argument. When you are proving a statement or theorem, you must use facts, definitions, and accepted properties in a logical order. You can also use previously proved theorems in your proof. For instance, the Distance Formula is used in the proof of the Midpoint Formula below. There are several different proof methods, which you will see in later chapters.

The Midpoint Formula *(p. 61)*

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is given by the Midpoint Formula

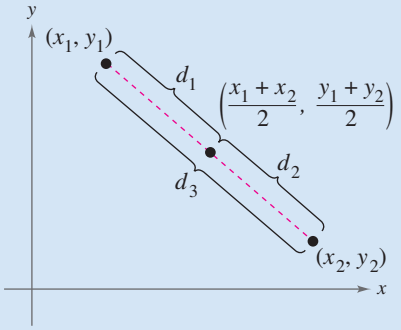
$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The Cartesian Plane

The Cartesian plane was named after the French mathematician René Descartes (1596–1650). While Descartes was lying in bed, he noticed a fly buzzing around on the square ceiling tiles. He discovered that the position of the fly could be described by which ceiling tile the fly landed on. This led to the development of the Cartesian plane. Descartes felt that a coordinate plane could be used to facilitate description of the positions of objects.

Proof

Using the figure, you must show that $d_1 = d_2$ and $d_1 + d_2 = d_3$.



By the Distance Formula, you obtain

$$d_1 = \sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d_2 = \sqrt{\left(x_2 - \frac{x_1 + x_2}{2}\right)^2 + \left(y_2 - \frac{y_1 + y_2}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

So, it follows that $d_1 = d_2$ and $d_1 + d_2 = d_3$.

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P.S. Problem Solving

This collection of thought-provoking and challenging exercises further explores and expands upon concepts learned in this chapter.

1. The NCAA states that the men's and women's shots for track and field competition must comply with the following specifications. (Source: NCAA)

	Men's	Women's
Weight (minimum)	7.26 kg	4.0 kg
Diameter (minimum)	110 mm	95 mm
Diameter (maximum)	130 mm	110 mm

(a) Find the maximum and minimum volumes of both the men's and women's shots.

(b) The *density* of an object is an indication of how heavy the object is. To find the density of an object, divide its mass (weight) by its volume. Find the maximum and minimum densities of both the men's and women's shots.

(c) A shot is usually made out of iron. If a ball of cork has the same volume as an iron shot, do you think they would have the same density? Explain your reasoning.

2. Find an example for which

$|a - b| > |a| - |b|$,

and an example for which

$|a - b| = |a| - |b|$.

Then prove that

$|a - b| \geq |a| - |b|$

for all a, b.

3. A major feature of Epcot Center at Disney World is called Spaceship Earth. The building is shaped as a sphere and weighs 1.6×10^7 pounds, which is equal in weight to 1.58×10^8 golf balls. Use these values to find the approximate weight (in pounds) of one golf ball. Then convert the weight to ounces. (Source: Disney.com)

4. The average life expectancies at birth in 2001 for men and women were 74.4 years and 79.8 years, respectively. Assuming an average healthy heart rate of 70 beats per minute, find the numbers of beats in a lifetime for a man and for a woman. (Source: National Center for Health Statistics)

5. The accuracy of an approximation to a number is related to how many significant digits there are in the approximation. Write a definition of significant digits and illustrate the concept with examples.

6. The table shows the census population y (in millions) of the United States for each census year x from 1950 to 2000. (Source: U.S. Census Bureau)

Year, x	Population, y
1950	151.33
1960	179.32
1970	203.30
1980	226.54
1990	248.72
2000	281.42

(a) Sketch a scatter plot of the data. Describe any trends in the data.

(b) Find the increase in population from each census year to the next.

(c) Over which decade did the population increase the most? the least?

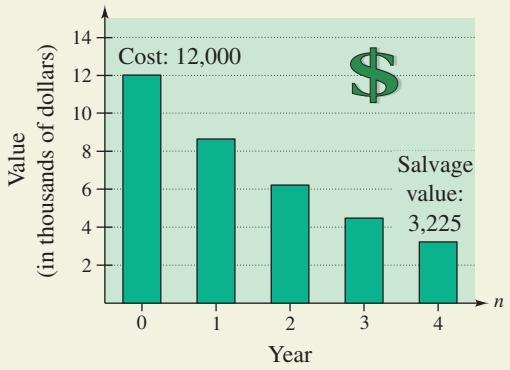
(d) Find the percent increase in population from each census year to the next.

(e) Over which decade was the percent increase the greatest? the least?

7. Find the annual depreciation rate r from the bar graph below. To find r by the declining balances method, use the formula

$$r = 1 - \left(\frac{S}{C}\right)^{1/n}$$

where n is the useful life of the item (in years), S is the salvage value (in dollars), and C is the original cost (in dollars).



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8. Johannes Kepler (1571–1630), a well-known German astronomer, discovered a relationship between the average distance of a planet from the sun and the time (or period) it takes the planet to orbit the sun. People then knew that planets that are closer to the sun take less time to complete an orbit than planets that are farther from the sun. Kepler discovered that the distance and period are related by an exact mathematical formula.

The table shows the average distances x (in astronomical units) and periods y (in years) for the five planets that are closest to the sun. By completing the table, can you rediscover Kepler's relationship? Write a paragraph that summarizes your conclusions.

Planet	Mercury	Venus	Earth	Mars	Jupiter
x	0.387	0.723	1.000	1.524	5.203
$\sqrt{x}$					
y	0.241	0.615	1.000	1.881	11.863
$\sqrt[3]{y}$					

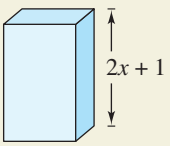
9. A stained glass window is designed in the shape of a rectangle with a semicircular arch (see figure). The width of the window is 2 feet and the perimeter is approximately 13.14 feet. Find the smallest amount of glass required to construct the window.



10. The volume V of the box (in cubic inches) shown in the figure is modeled by

$$V = 2x^3 + x^2 - 8x - 4$$

where x is measured in inches. Find an expression for the surface area of the box. Then find the surface area when $x = 6$ inches.



11. Verify that $y_1 \neq y_2$ by letting $x = 0$ and evaluating y_1 and y_2.

$$y_1 = 2x\sqrt{1 - x^2} - \frac{x^3}{\sqrt{1 - x^2}}$$

$$y_2 = \frac{2 - 3x^2}{\sqrt{1 - x^2}}$$

Change y_2 so that $y_1 = y_2$.

12. Prove that

$$\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3}\right)$$

is one of the points of trisection of the line segment joining (x_1, y_1) and (x_2, y_2). Find the midpoint of the line segment joining

$$\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3}\right)$$

and (x_2, y_2) to find the second point of trisection.

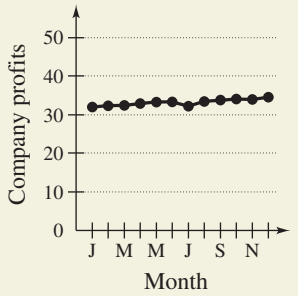
13. Use the results of Exercise 12 to find the points of trisection of the line segment joining each pair of points.

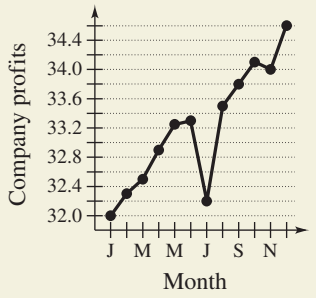
(a) $(1, -2), (4, 1)$ (b) $(-2, -3), (0, 0)$

14. Although graphs can help visualize relationships between two variables, they can also be used to mislead people. The graphs shown below represent the same data points.

(a) Which of the two graphs is misleading, and why? Discuss other ways in which graphs can be misleading.

(b) Why would it be beneficial for someone to use a misleading graph?





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